

Finite Difference Time Domain Approach

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Sources:

- Computational Electrodynamics: The FDTD Method - A Taflove & S. Hagness
- Electromagnetic Simulation Using the FDTD Method - D.M. Sullivan
- Course notes on Electromagnetic analysis using FDTD - EMLAB

1st - preliminary result : BPM

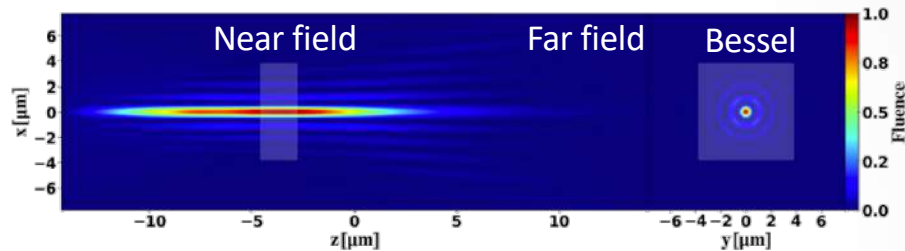
Gaussian beam



Axicon lens



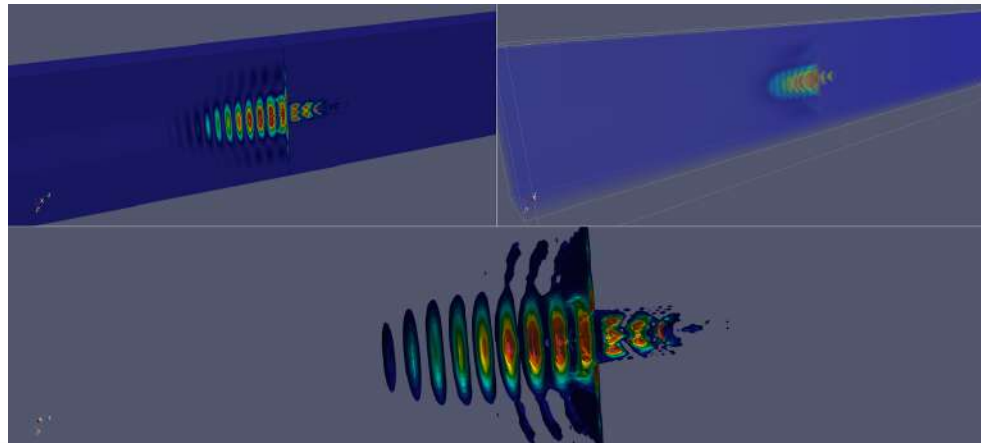
Nondiffractive beams



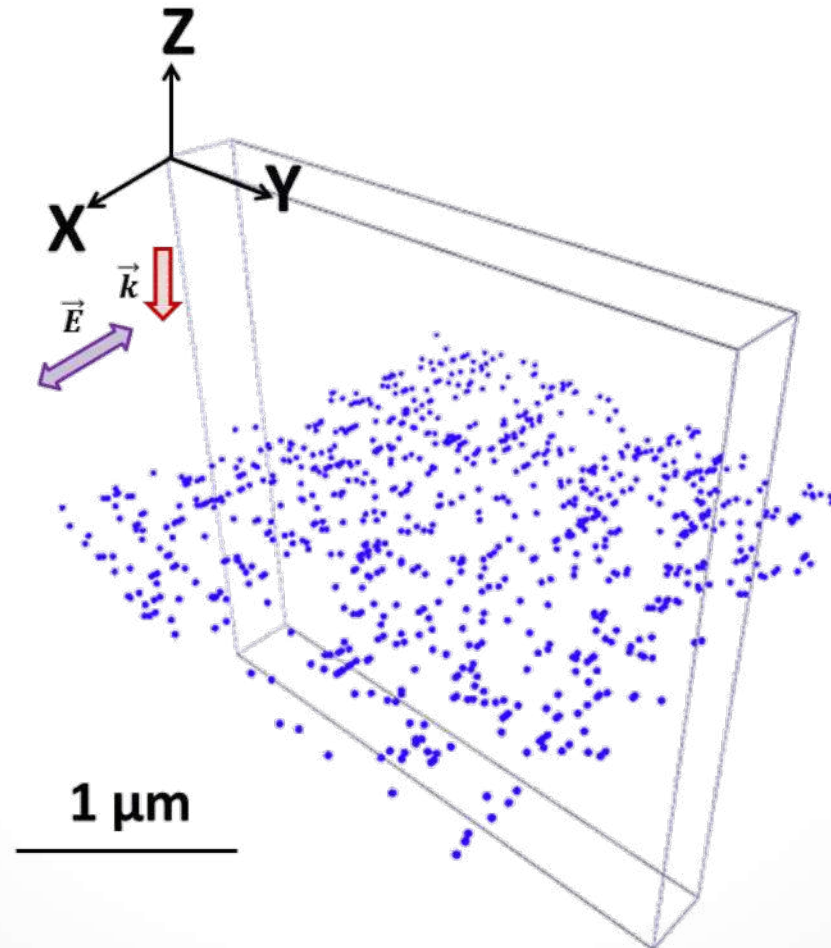
Polarization
at the focus?

Next step: vectorial and 3D
[Maxwell Equations]

$$\begin{cases} \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} - \vec{J}_{Lin} - \vec{J}_{NL} \\ \frac{\partial \vec{H}}{\partial t} = -\frac{\nabla \times \vec{E}}{\mu_0} \end{cases}$$



- Example of light propagation and surface interaction

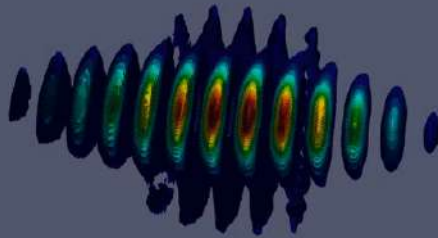


Nonlinear 3D propagation by solving Maxwell equations (FDTD approach)

Pulsed laser beam

Interaction

Material modifications



Control the energy distribution



Light coupling

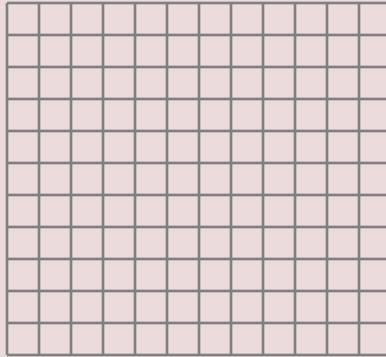
Surface/volume functionalization



Polarization effects
Interference effects
Scattering phenomena

...

Finite differences

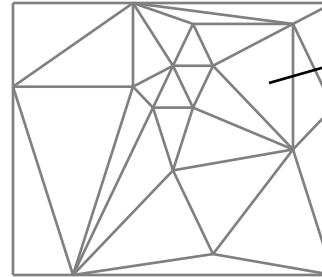


Discrete
unknown
values

On a regular
grid

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{\Delta x} + O(\Delta x^2)$$

Finite elements



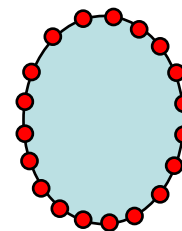
Unknown values
approximated by polynomial
functions in irregular
elements

Spectral methods

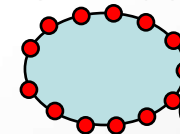


Decomposition on a
complete base
(as Fourier series)

Boundary element methods

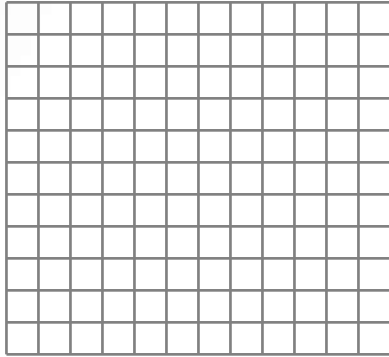


Solving the partial differential
equations formulated as
integral equations
(i.e. in boundary integral form)

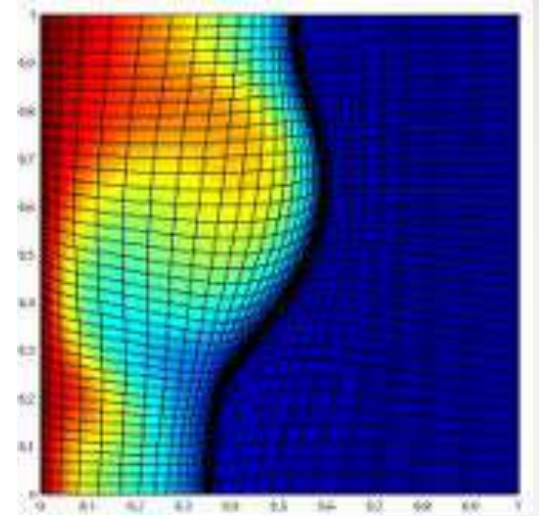


Green functions

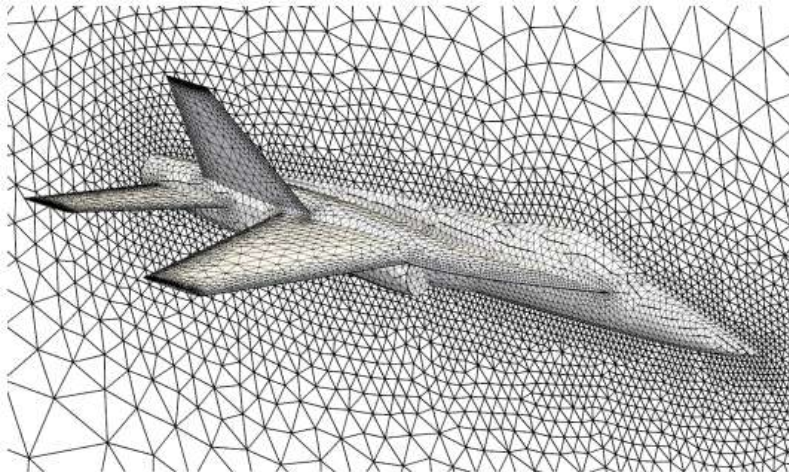
The grid can be considered...



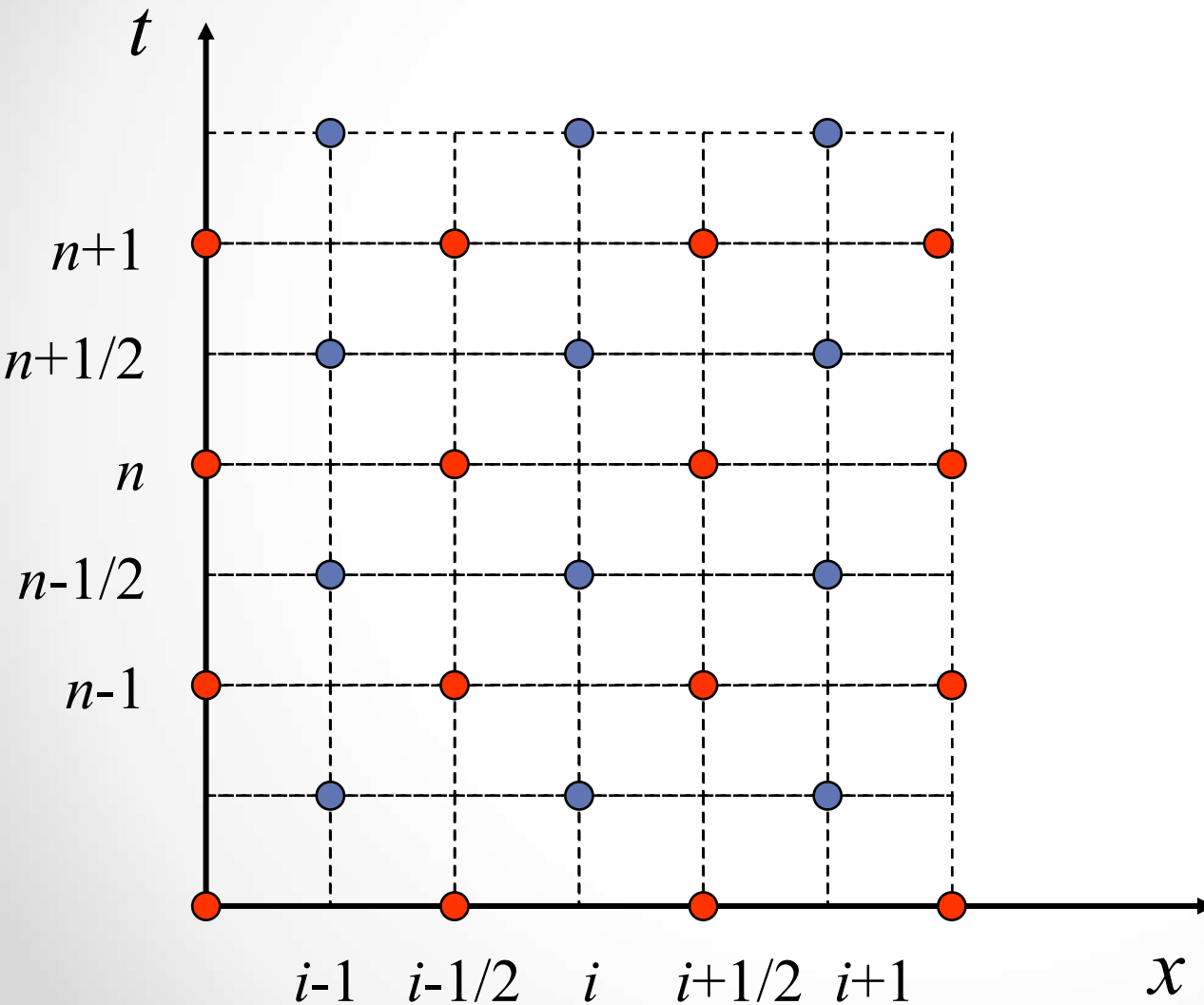
Simple



Deformable



or just complex...

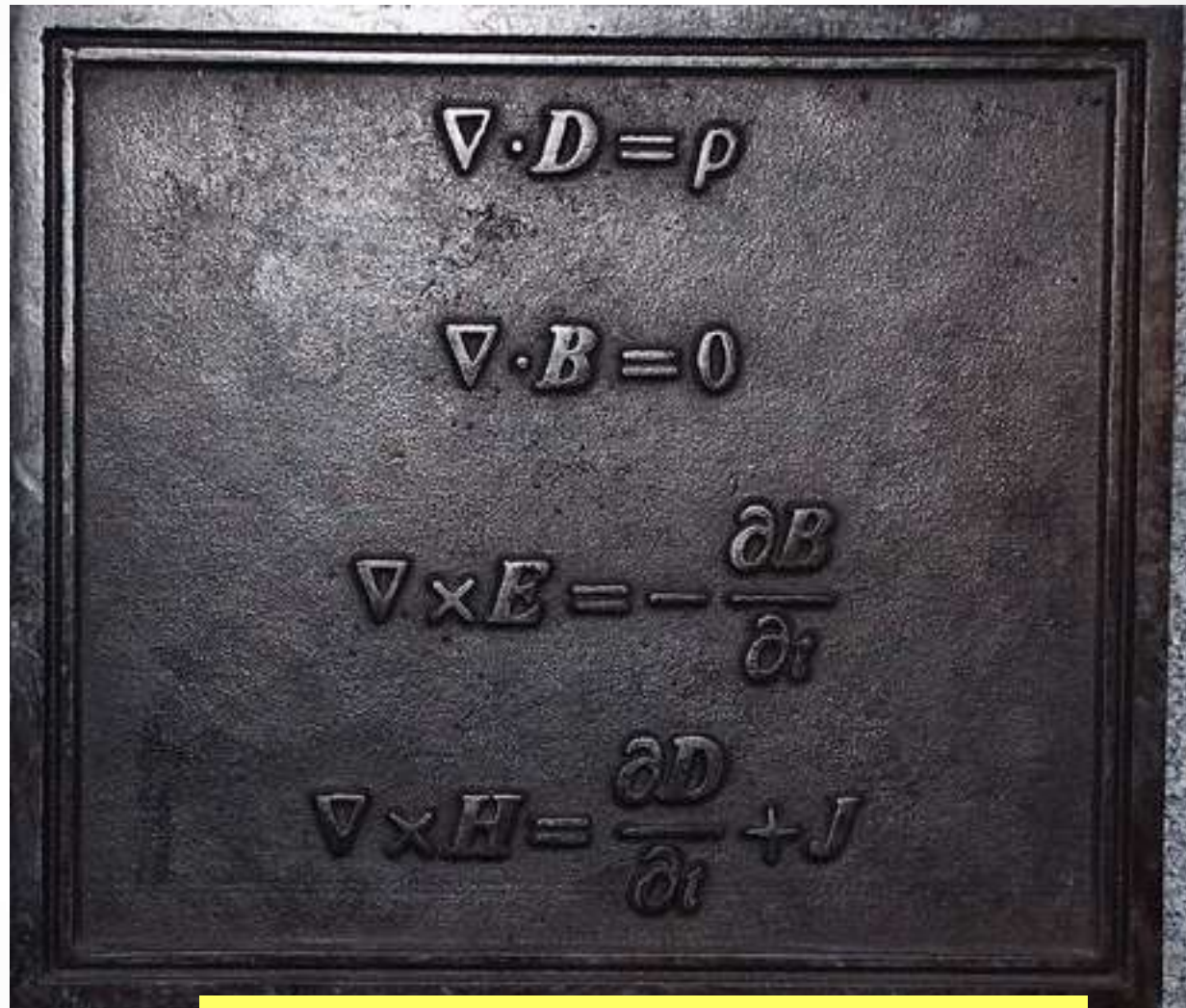


$\vec{E}$
$\vec{P}$
$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$
$\vec{H}$
$\vec{B} = \mu_0 \vec{H}$
$\vec{\Pi} = \vec{E} \times \vec{H}$

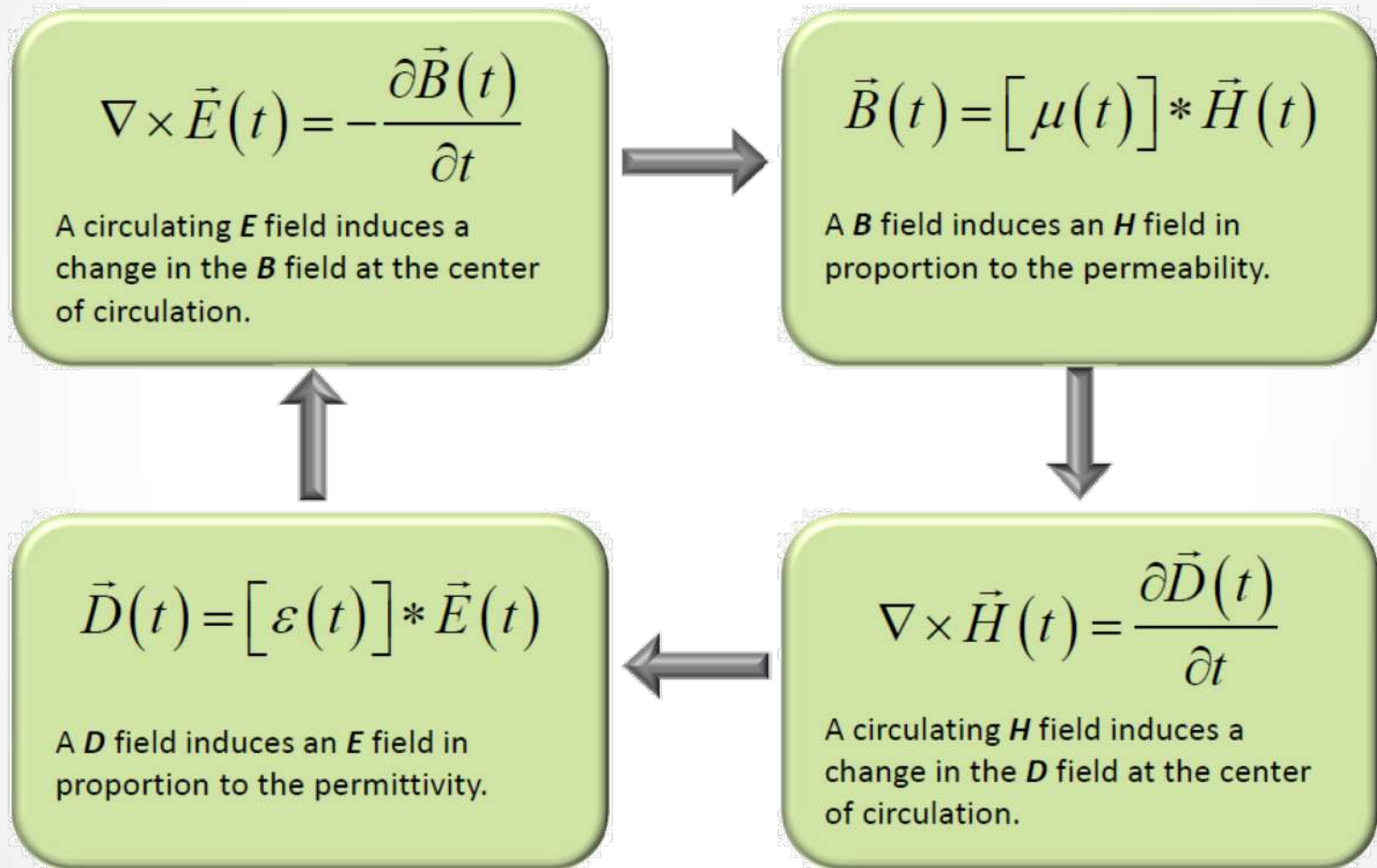
● E field

● H field

Maxwell equations



Engraved in stone...



Note: In reality, this all happens simultaneously. In FDTD, it follows this flow.

In materials that are linear, isotropic and non-dispersive we have

$$[\mu(t)]^* = \mu \cdot \quad [\varepsilon(t)]^* = \varepsilon \cdot$$

In this case, the flow of Maxwell's equations reduces to

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t}$$

A circulating E field induces a change in the H field at the center of circulation in proportion to the permeability.



$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t}$$

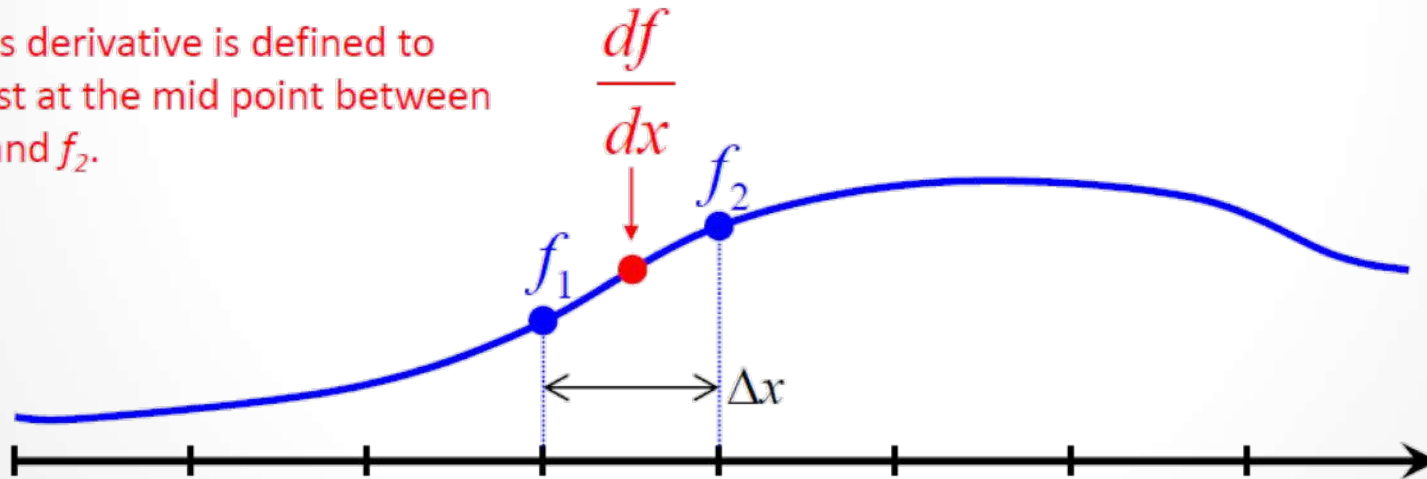
A circulating H field induces a change in the E field at the center of circulation in proportion to the permittivity.



$$\frac{df_{1.5}}{dx} \approx \frac{f_2 - f_1}{\Delta x}$$

second-order accurate
first-order derivative

This derivative is defined to exist at the mid point between f_1 and f_2 .



Each term in a finite-difference equation must exist at the same point in time and space.

Example: $\frac{\partial f(x)}{\partial x} + f(x) = 0$ Given $f(0), f(\Delta x), f(2\Delta x), \dots$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} + f(x) = 0$$

Exists at $x + \Delta x/2$ Exists at x

Your simulation will be unstable (i.e. explode).

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} + f\left(x + \frac{\Delta x}{2}\right) = 0$$

Exists at $x + \Delta x/2$ Exists at $x + \Delta x/2$

$f(x)$ is only known at integer multiples of Δx . How do we calculate $f(x + \Delta x/2)$?

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{f(x + \Delta x) + f(x)}{2} = 0$$

Exists at $x + \Delta x/2$ Exists at $x + \Delta x/2$

An intuitive first guess at approximating the time derivatives in Maxwell's equations is:

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t} \quad \Rightarrow \quad \boxed{\nabla \times \vec{E}(t)} \cong -\mu \frac{\boxed{\vec{H}(t + \Delta t) - \vec{H}(t)}}{\Delta t}$$

Exists at t
Exists at $t + \Delta t/2$

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t} \quad \Rightarrow \quad \boxed{\nabla \times \vec{H}(t)} \cong \varepsilon \frac{\boxed{\vec{E}(t + \Delta t) - \vec{E}(t)}}{\Delta t}$$

Exists at t
Exists at $t + \Delta t/2$

This is an unstable formulation.

We adjust the finite-difference equations so that each term exists at the same point in time.

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{E}(t) = -\mu \frac{\vec{H}(t + \Delta t/2) - \vec{H}(t - \Delta t/2)}{\Delta t}$$

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{H}(t + \Delta t/2) = \varepsilon \frac{\vec{E}(t + \Delta t) - \vec{E}(t)}{\Delta t}$$

These equations will get messy if we include interpolations.

Is there a simpler approach?

We stagger **E** and **H** in time so that **E** exists at integer time steps $(0, \Delta t, 2\Delta t, \dots)$ and **H** exists at half time steps $(\Delta t/2, t+\Delta t/2, 2t+\Delta t/2, \dots)$.

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{E} \Big|_t = -\mu \frac{\vec{H} \Big|_{t+\Delta t/2} - \vec{H} \Big|_{t-\Delta t/2}}{\Delta t}$$

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{H} \Big|_{t+\Delta t/2} = \varepsilon \frac{\vec{E} \Big|_{t+\Delta t} - \vec{E} \Big|_t}{\Delta t}$$

 We will handle the spatial derivatives in $\nabla \times$ next lecture in a very similar manner.

The “update equations” are the equations used inside the main FDTD loop to calculate the field values at the next time step.

They are derived by solving our finite-difference equations for the fields at the future time values.

$$\nabla \times \vec{E} \Big|_t = -\mu \frac{\vec{H} \Big|_{t+\Delta t/2} - \vec{H} \Big|_{t-\Delta t/2}}{\Delta t} \quad \Rightarrow \quad \vec{H} \Big|_{t+\Delta t/2} = \vec{H} \Big|_{t-\Delta t/2} - \frac{\Delta t}{\mu} \left(\nabla \times \vec{E} \Big|_t \right)$$

$$\nabla \times \vec{H} \Big|_{t+\Delta t/2} = \varepsilon \frac{\vec{E} \Big|_{t+\Delta t} - \vec{E} \Big|_t}{\Delta t} \quad \Rightarrow \quad \vec{E} \Big|_{t+\Delta t} = \vec{E} \Big|_t + \frac{\Delta t}{\varepsilon} \left(\nabla \times \vec{H} \Big|_{t+\Delta t/2} \right)$$

Update coefficient

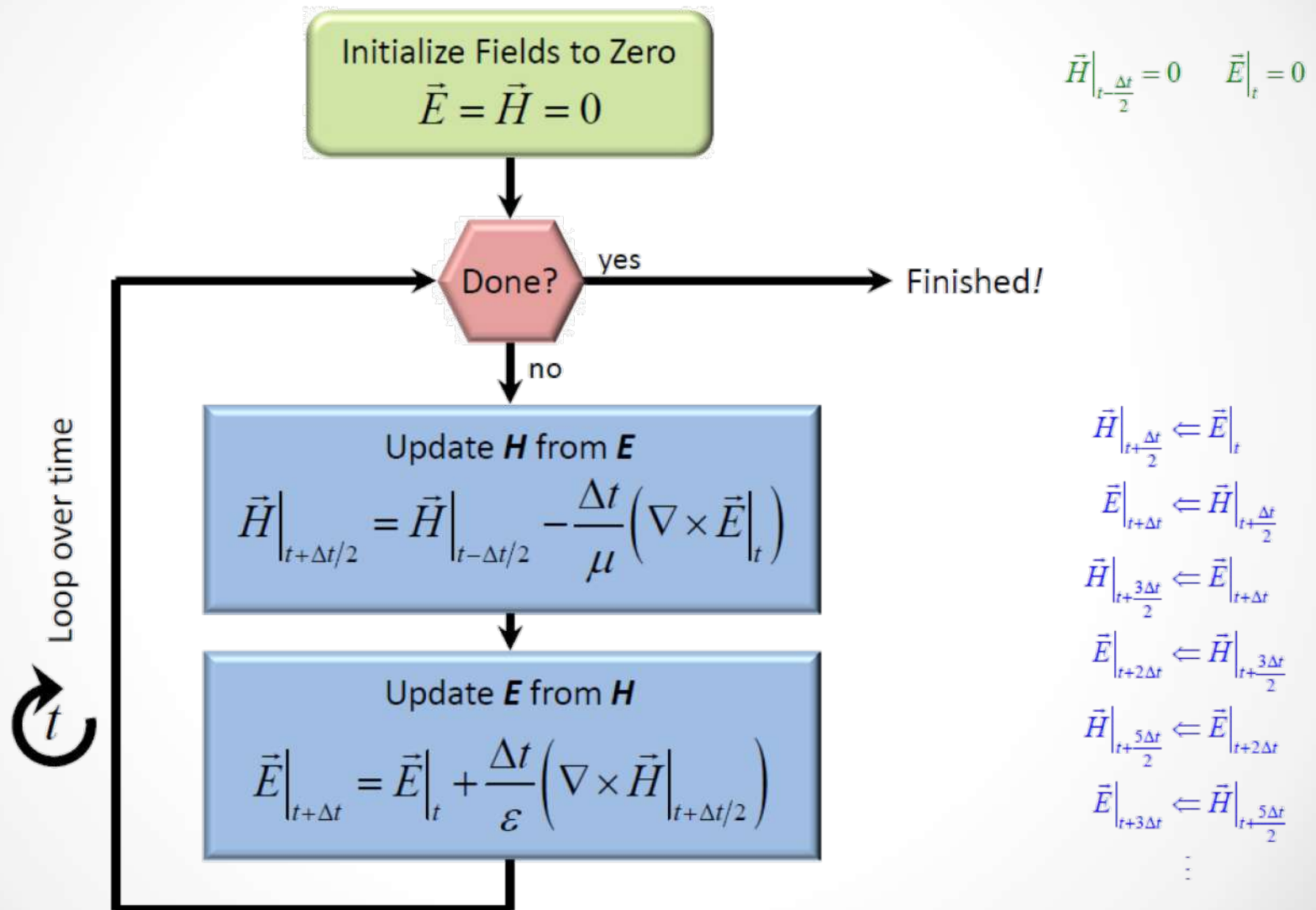
To speed simulation, we calculate these before iteration.

$$\vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_t + \frac{\Delta t}{\epsilon} \left(\nabla \times \vec{H}\Big|_{t+\Delta t/2} \right)$$

Field at the future time step.

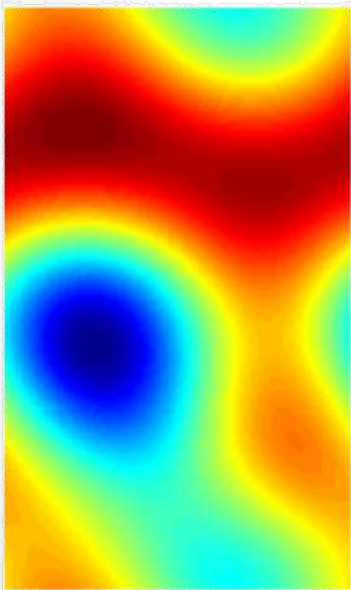
Field at the previous time step.

Curl of the “other” field at an intermediate time step

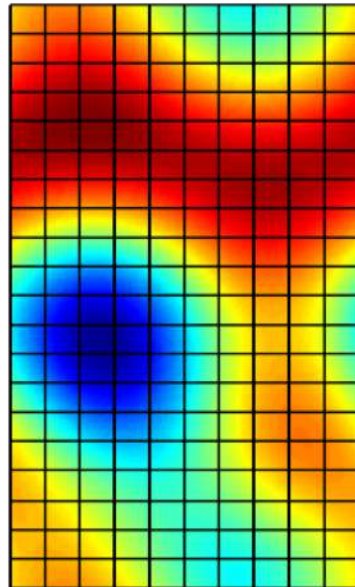


Functions on a grid

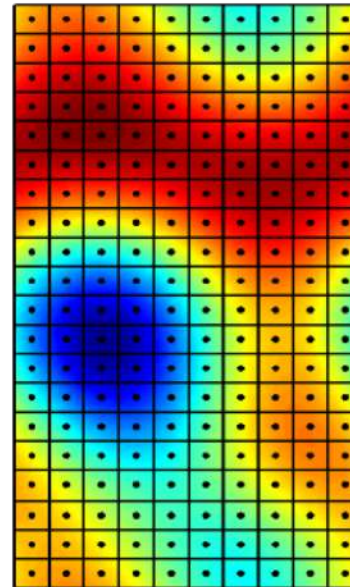
Example
physical
(continuous)
2D function



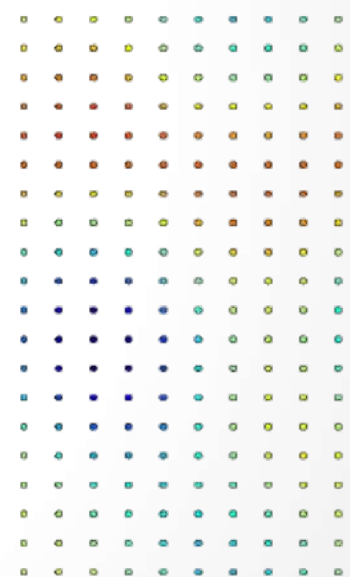
A grid is
constructed by
dividing space
into discrete
cells

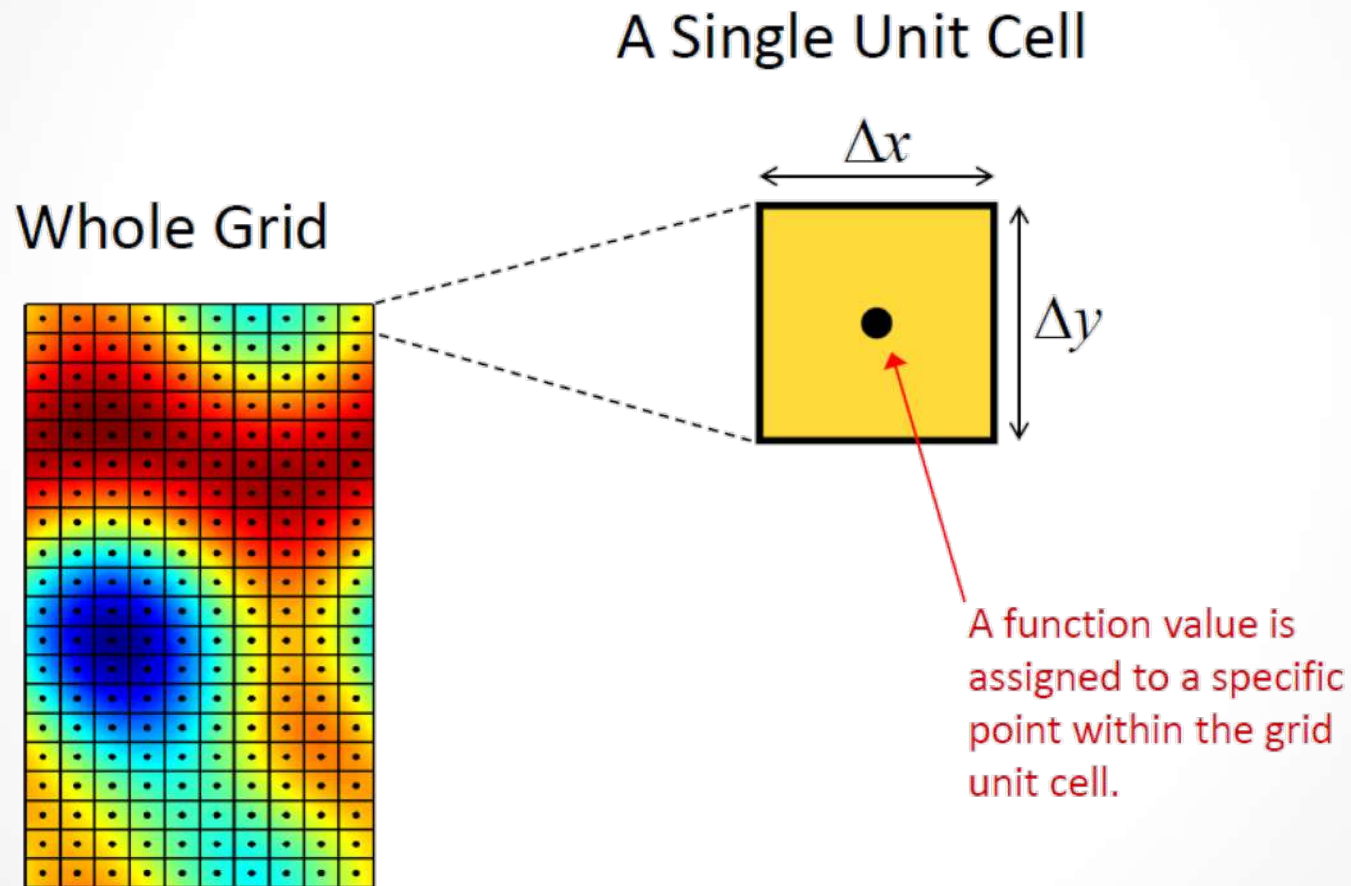


Function is
known only at
discrete points

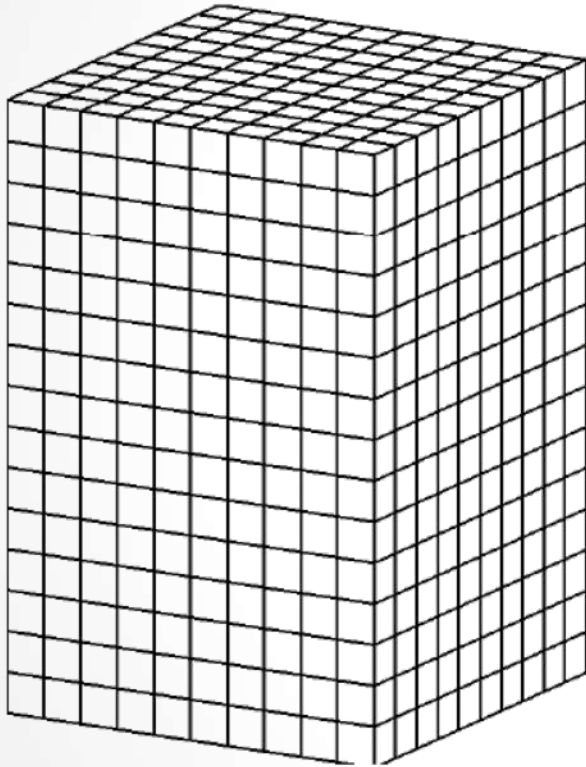


Representation
of what is
actually stored in
memory



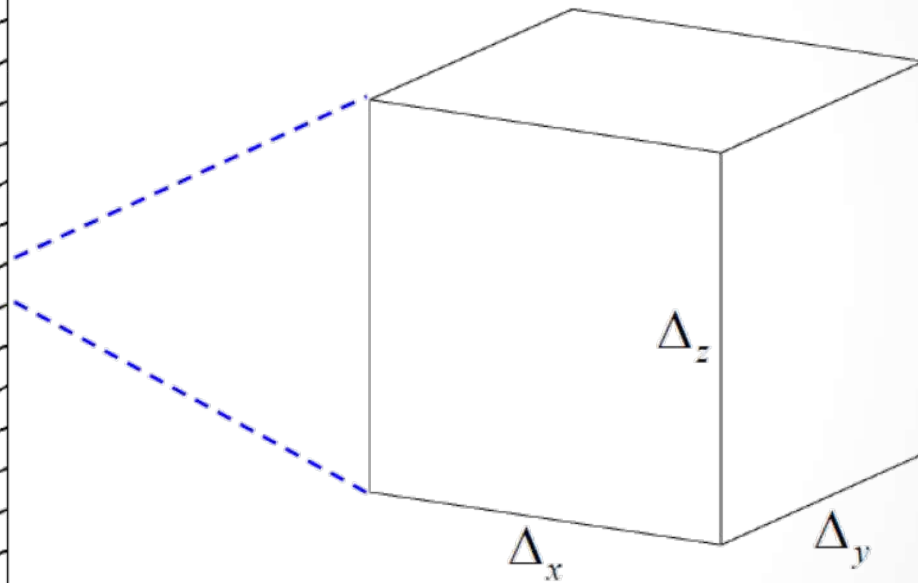


A three-dimensional grid looks like this:

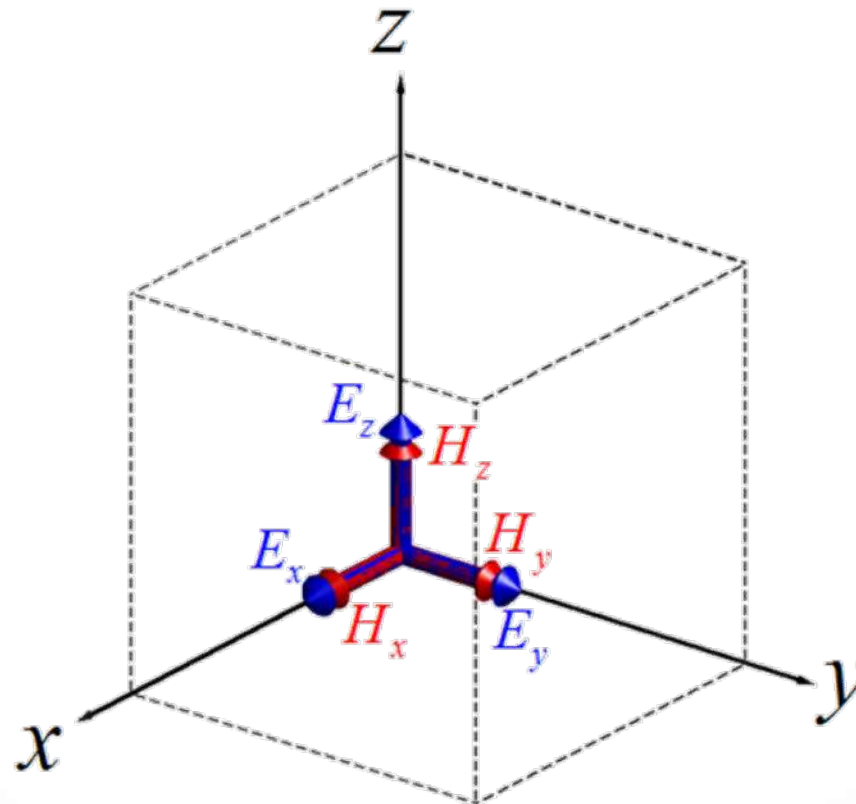


$$N_x = 10, N_y = 10, N_z = 15$$

A unit cell from the grid looks like this:

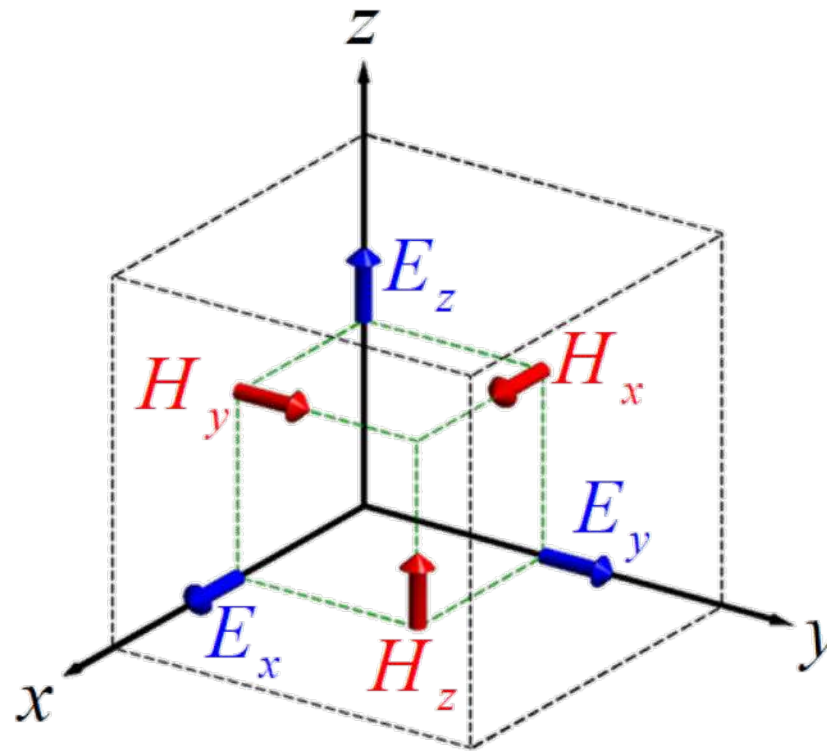


Within the unit cell, we need to place the field components E_x , E_y , E_z , H_x , H_y , and H_z .



A straightforward approach would be to locate all of the field components within in a grid cell at the origin of the cell.

Instead, we are going to stagger the position of each field component within the grid cells.



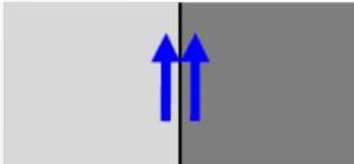
K. S. Yee, "Numerical solution of the initial boundary value problems involving Maxwell's equations in isotropic media," IEEE Trans. Microwave Theory and Techniques, vol. 44, pp. 61–69, 1998.

1. Divergence-free

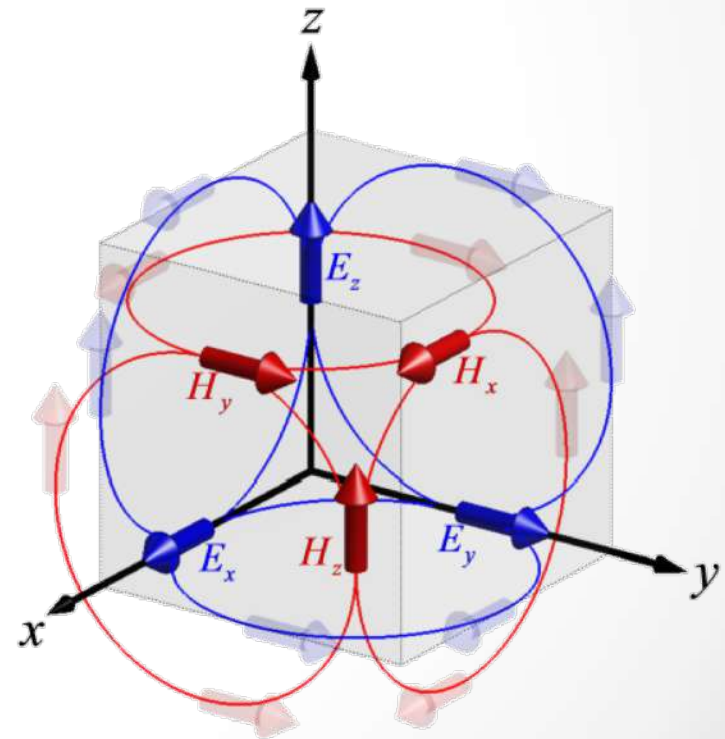
$$\nabla \cdot (\epsilon \vec{E}) = 0$$

$$\nabla \cdot (\mu \vec{H}) = 0$$

2. Physical boundary conditions are naturally satisfied

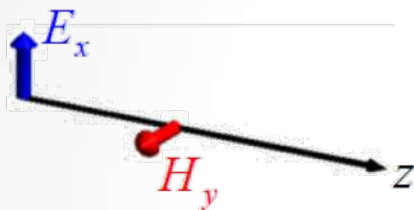


3. Elegant arrangement to approximate Maxwell's curl equations



Yee cell for 1D, 2D, and 3D grids

1D Yee Grid

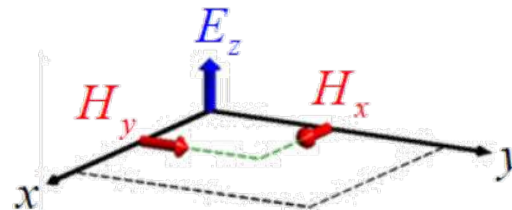


E_x Mode

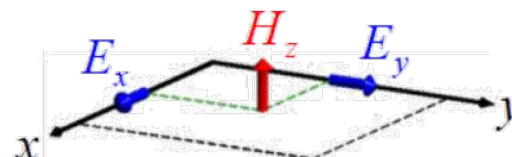


E_y Mode

2D Yee Grids

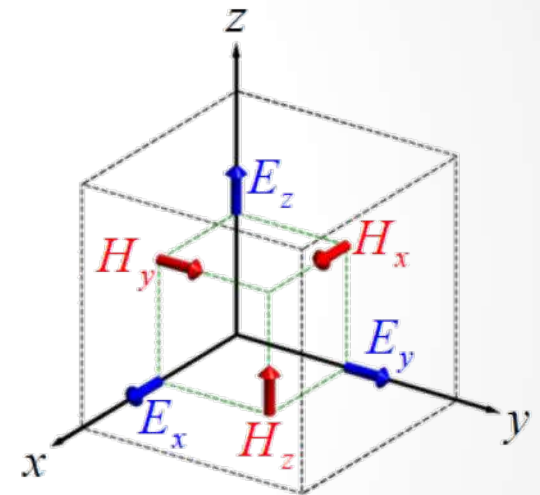


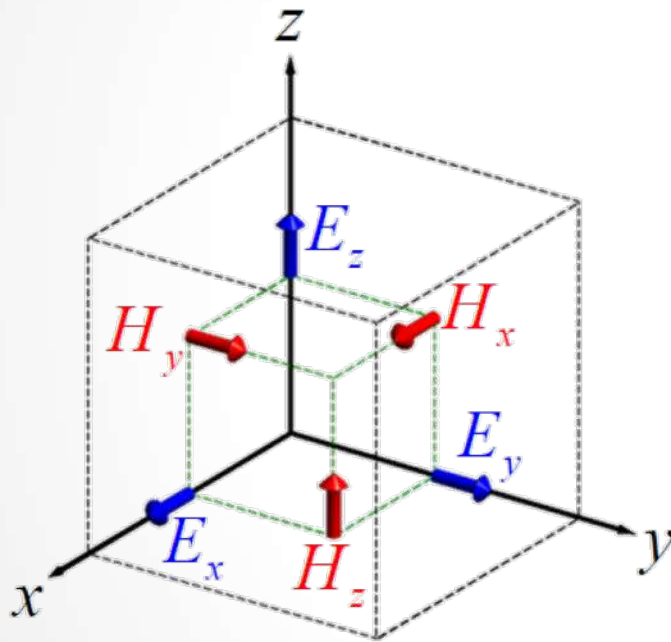
E_z Mode



H_z Mode

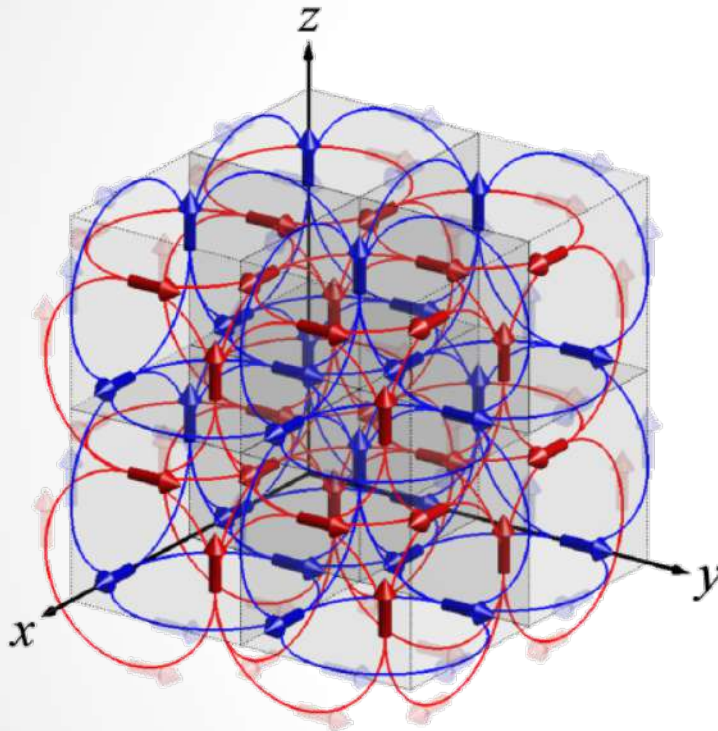
3D Yee Grid



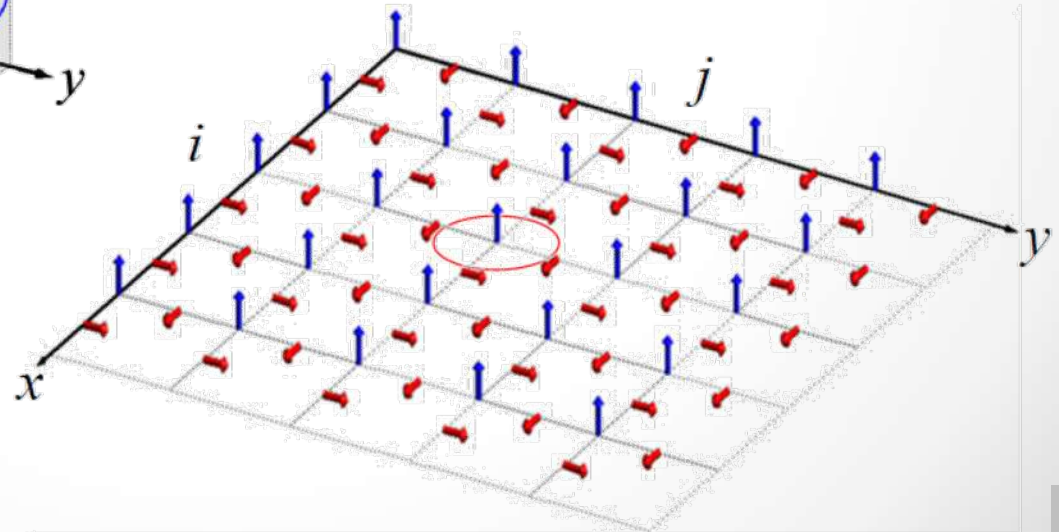


- Field components are in physically different locations
- Field components may reside in different materials even if they are in the same unit cell
- Field components will be out of phase
- Recall the field components are also staggered in time.

2×2×2 Grid



4×4 Grid (E Mode)



The impedance of a material quantifies the relation between the electric and magnetic field of a wave travelling through that material. It is the most fundamental quantity that causes reflections and scattering.

$$\eta \simeq |\vec{E}|/|\vec{H}|$$

The impedance can be written relative to the free space impedance as

$$\eta = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.73031346177 \ \Omega$$

This shows that the electric field is around two to three orders of magnitude larger than the magnetic field.

We satisfied the divergence equations by adopting the Yee grid scheme. We now only have to deal with the curl equations.

$$\nabla \times \vec{E} = -[\mu] \frac{\partial \vec{H}}{\partial t} \qquad \nabla \times \vec{H} = [\varepsilon] \frac{\partial \vec{E}}{\partial t}$$

The $\vec{E}$ and $\vec{H}$ fields are related through the impedance of the material they are in, so they are roughly three orders of magnitude different.

$$|\vec{E}| \cong \eta |\vec{H}| \qquad \eta \sim 300 \, \Omega$$

This will cause rounding errors in your simulation and it is always good practice to normalize your parameters so they are all the same order of magnitude. Here we choose to normalize the magnetic field.

$$|\vec{E}| \cong |\vec{\tilde{H}}| \quad \rightarrow \quad \vec{\tilde{H}} = \eta \vec{H} \qquad \boxed{\vec{\tilde{H}} = \eta_0 \vec{H}}$$

We do not know the actual impedance ahead of time, so we just use the free space impedance.

Using the normalized magnetic field, the curl equations become

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t} \qquad \nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t}$$

Proof

$$\vec{\tilde{H}} = \eta_0 \vec{H} \rightarrow \vec{H} = \vec{\tilde{H}} / \eta_0$$

$$\nabla \times \vec{E} = -[\mu] \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{E} = -[\mu_0 \mu_r] \frac{\partial (\vec{\tilde{H}} / \eta_0)}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\mu_0}{\eta_0} [\mu_r] \frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} [\mu_r] \frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\sqrt{\mu_0 \varepsilon_0} [\mu_r] \frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{H} = [\varepsilon] \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\vec{\tilde{H}} / \eta_0) = [\varepsilon_0 \varepsilon_r] \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \varepsilon_0 \eta_0 [\varepsilon_r] \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \varepsilon_0 \sqrt{\frac{\mu_0}{\varepsilon_0}} [\varepsilon_r] \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \sqrt{\mu_0 \varepsilon_0} [\varepsilon_r] \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t}$$

Note:

$$[\mu] = \mu_0 [\mu_r]$$

$$[\varepsilon] = \varepsilon_0 [\varepsilon_r]$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Expand the curl equations

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t}$$



$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{1}{c_0} \left(\mu_{xx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{xy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{xz} \frac{\partial \tilde{H}_z}{\partial t} \right) \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{1}{c_0} \left(\mu_{yx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{yy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{yz} \frac{\partial \tilde{H}_z}{\partial t} \right) \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{1}{c_0} \left(\mu_{zx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{zy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{zz} \frac{\partial \tilde{H}_z}{\partial t} \right) \end{aligned}$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\epsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t}$$



$$\begin{aligned} \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= \frac{1}{c_0} \left(\epsilon_{xx} \frac{\partial E_x}{\partial t} + \epsilon_{xy} \frac{\partial E_y}{\partial t} + \epsilon_{xz} \frac{\partial E_z}{\partial t} \right) \\ \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} &= \frac{1}{c_0} \left(\epsilon_{yx} \frac{\partial E_x}{\partial t} + \epsilon_{yy} \frac{\partial E_y}{\partial t} + \epsilon_{yz} \frac{\partial E_z}{\partial t} \right) \\ \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} &= \frac{1}{c_0} \left(\epsilon_{zx} \frac{\partial E_x}{\partial t} + \epsilon_{zy} \frac{\partial E_y}{\partial t} + \epsilon_{zz} \frac{\partial E_z}{\partial t} \right) \end{aligned}$$

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0} \frac{\partial \vec{H}}{\partial t}$$



$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c_0} \left(\mu_{xx} \frac{\partial \tilde{H}_x}{\partial t} + \cancel{\mu_{xy} \frac{\partial \tilde{H}_y}{\partial t}} + \cancel{\mu_{xz} \frac{\partial \tilde{H}_z}{\partial t}} \right)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c_0} \left(\cancel{\mu_{yx} \frac{\partial \tilde{H}_x}{\partial t}} + \mu_{yy} \frac{\partial \tilde{H}_y}{\partial t} + \cancel{\mu_{yz} \frac{\partial \tilde{H}_z}{\partial t}} \right)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c_0} \left(\cancel{\mu_{zx} \frac{\partial \tilde{H}_x}{\partial t}} + \cancel{\mu_{zy} \frac{\partial \tilde{H}_y}{\partial t}} + \mu_{zz} \frac{\partial \tilde{H}_z}{\partial t} \right)$$

$$\nabla \times \vec{H} = \frac{[\epsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t}$$



$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{1}{c_0} \left(\epsilon_{xx} \frac{\partial E_x}{\partial t} + \cancel{\epsilon_{xy} \frac{\partial E_y}{\partial t}} + \cancel{\epsilon_{xz} \frac{\partial E_z}{\partial t}} \right)$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{1}{c_0} \left(\cancel{\epsilon_{yx} \frac{\partial E_x}{\partial t}} + \epsilon_{yy} \frac{\partial E_y}{\partial t} + \cancel{\epsilon_{yz} \frac{\partial E_z}{\partial t}} \right)$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{1}{c_0} \left(\cancel{\epsilon_{zx} \frac{\partial E_x}{\partial t}} + \cancel{\epsilon_{zy} \frac{\partial E_y}{\partial t}} + \epsilon_{zz} \frac{\partial E_z}{\partial t} \right)$$

These are the final form of Maxwell's equations from which we will formulate the FDTD method.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}$$

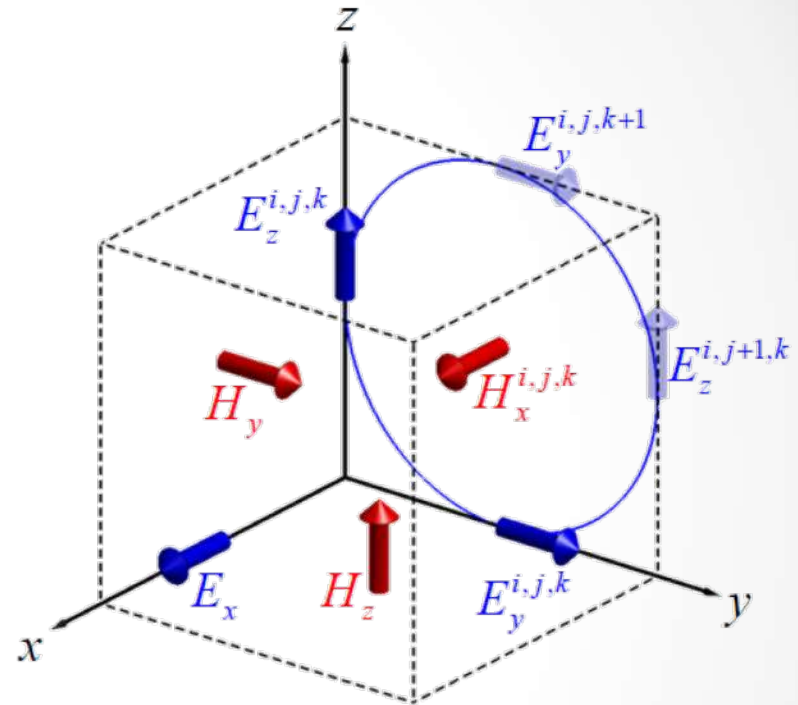
$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t}$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t}$$

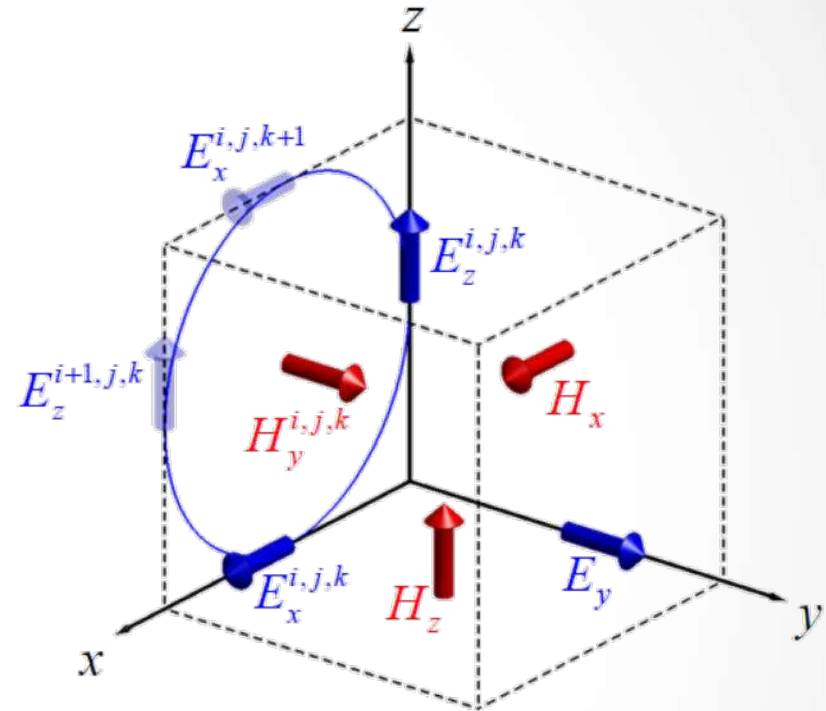
Next, we will approximate these equations with finite-differences in the Yee grid.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t}$$



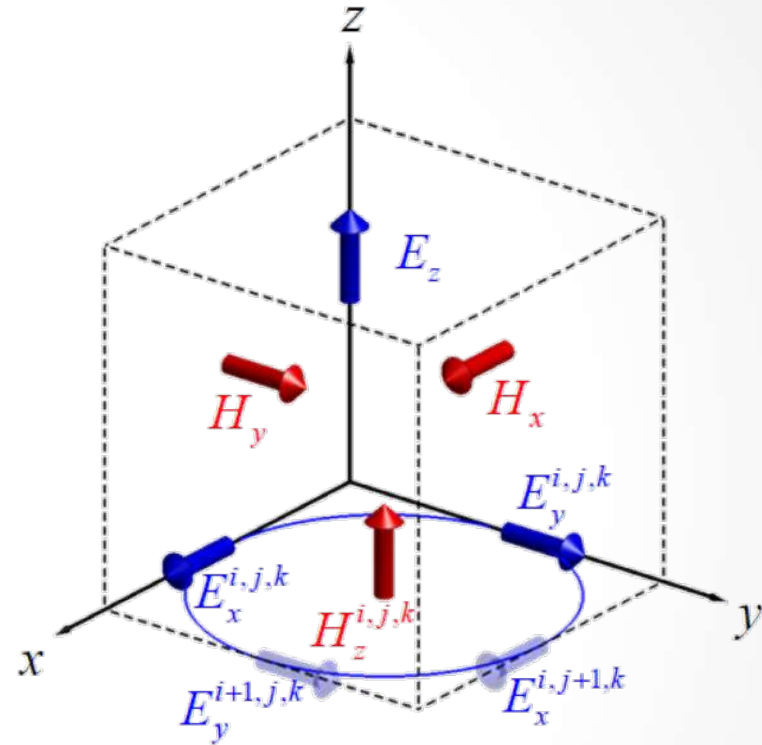
$$\frac{E_z^{i,j+1,k} \Big|_t - E_z^{i,j,k} \Big|_t}{\Delta y} - \frac{E_y^{i,j,k+1} \Big|_t - E_y^{i,j,k} \Big|_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k} \Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t}$$

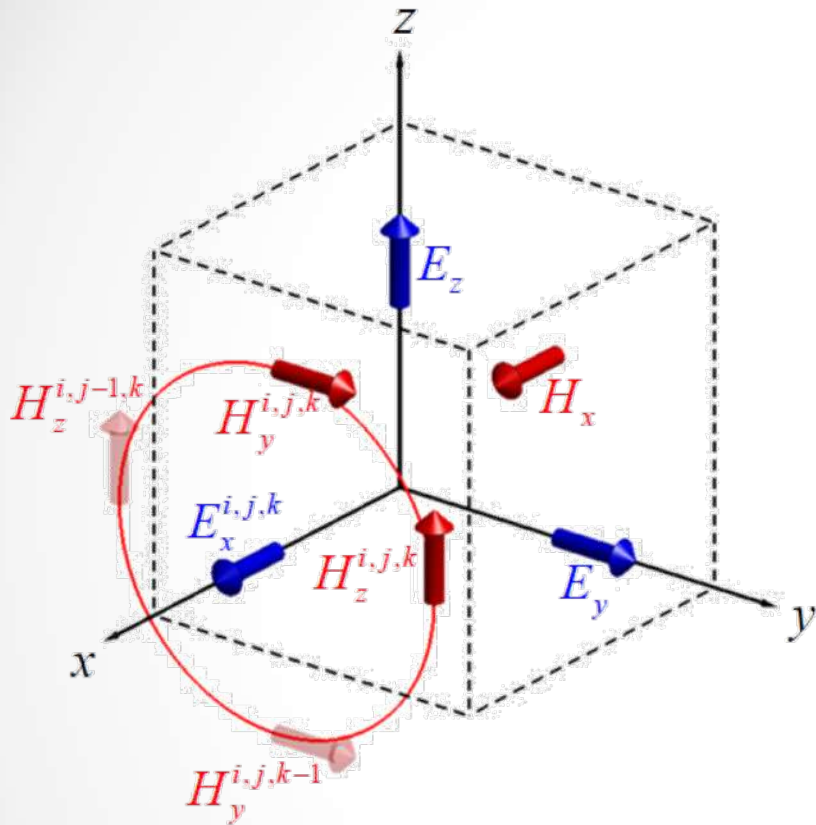


$$\frac{E_x^{i,j,k+1} \Big|_t - E_x^{i,j,k} \Big|_t}{\Delta z} - \frac{E_z^{i+1,j,k} \Big|_t - E_z^{i,j,k} \Big|_t}{\Delta x} = -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k} \Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}$$

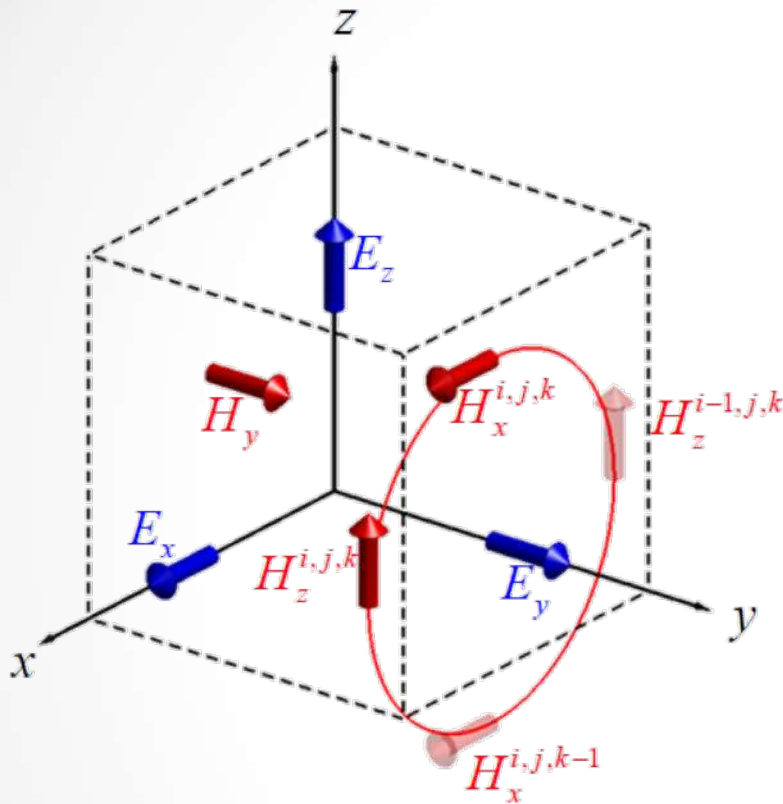


$$\frac{E_y^{i+1,j,k} \Big|_t - E_y^{i,j,k} \Big|_t}{\Delta x} - \frac{E_x^{i,j+1,k} \Big|_t - E_x^{i,j,k} \Big|_t}{\Delta y} = -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k} \Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$



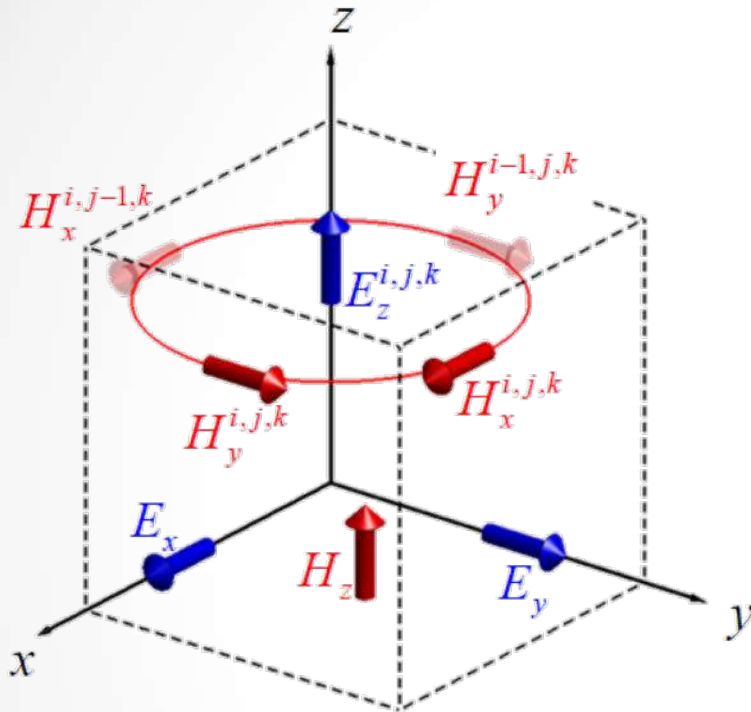
$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t}$$

$$\frac{\tilde{H}_z^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k} \Big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} \Big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} \Big|_{t+\Delta t} - E_x^{i,j,k} \Big|_t}{\Delta t}$$



$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t}$$

$$\frac{\tilde{H}_x^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} \Big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k} \Big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} \Big|_{t+\Delta t} - E_y^{i,j,k} \Big|_t}{\Delta t}$$



$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t}$$

$$\frac{\tilde{H}_y^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k} \Big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k} \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k} \Big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k} \Big|_{t+\Delta t} - E_z^{i,j,k} \Big|_t}{\Delta t}$$

$$\begin{aligned}\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}\end{aligned}$$



$$\begin{aligned}\frac{E_z^{i,j+1,k}|_t - E_z^{i,j,k}|_t}{\Delta y} - \frac{E_y^{i,j,k+1}|_t - E_y^{i,j,k}|_t}{\Delta z} &= -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \\ \frac{E_x^{i,j,k+1}|_t - E_x^{i,j,k}|_t}{\Delta z} - \frac{E_z^{i+1,j,k}|_t - E_z^{i,j,k}|_t}{\Delta x} &= -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \\ \frac{E_y^{i+1,j,k}|_t - E_y^{i,j,k}|_t}{\Delta x} - \frac{E_x^{i,j+1,k}|_t - E_x^{i,j,k}|_t}{\Delta y} &= -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t}\end{aligned}$$

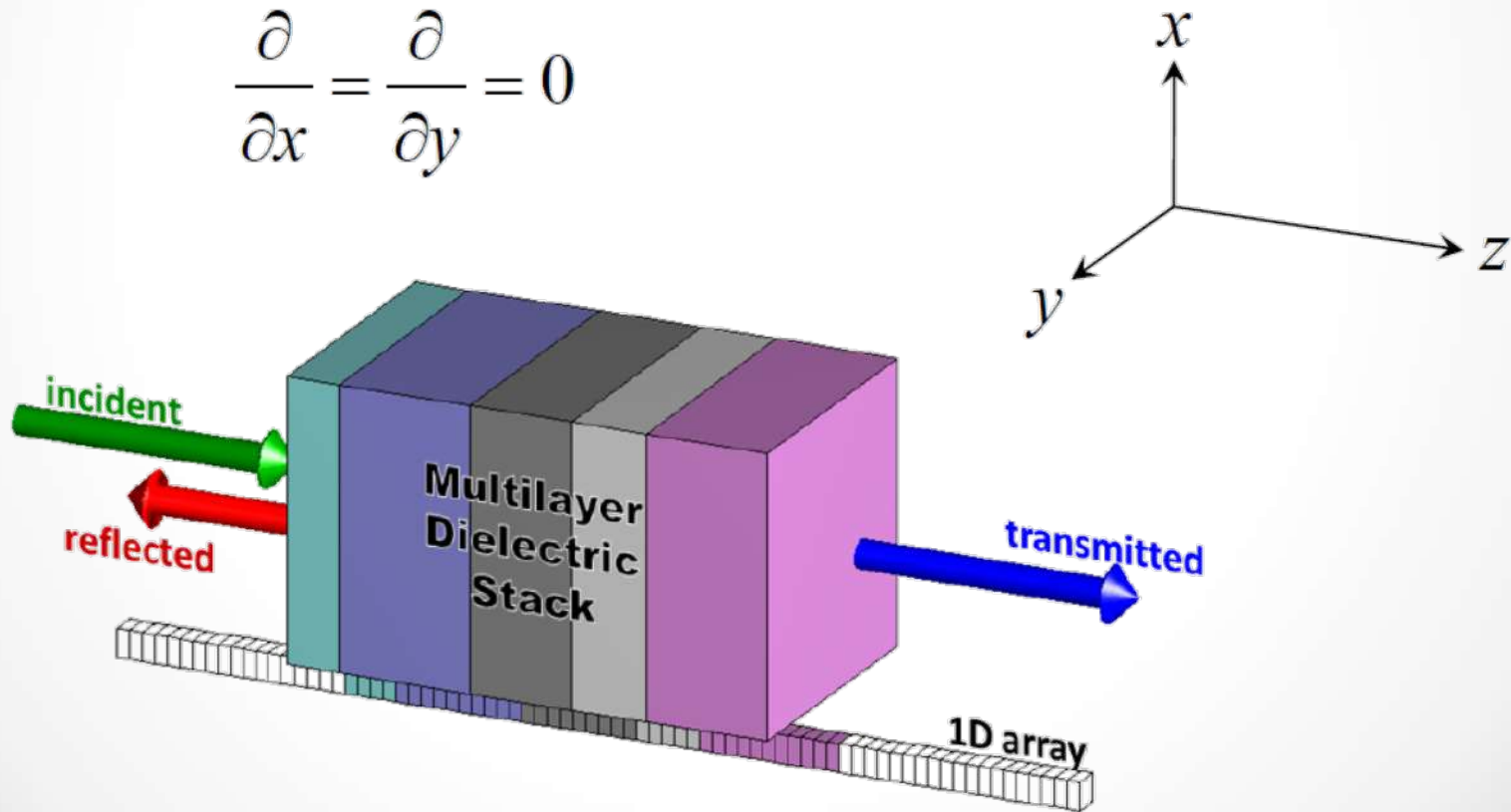
$$\begin{aligned}\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= \frac{\varepsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \\ \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} &= \frac{\varepsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \\ \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} &= \frac{\varepsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k}|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\varepsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k}|_{t+\Delta t} - E_x^{i,j,k}|_t}{\Delta t} \\ \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k}|_{t+\frac{\Delta t}{2}}}{\Delta x} &= \frac{\varepsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k}|_{t+\Delta t} - E_y^{i,j,k}|_t}{\Delta t} \\ \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k}|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k}|_{t+\frac{\Delta t}{2}}}{\Delta y} &= \frac{\varepsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k}|_{t+\Delta t} - E_z^{i,j,k}|_t}{\Delta t}\end{aligned}$$

Each equation is enforced separately for each cell in the grid. This is repeated for each time step until the simulation is finished. These equations get repeated a lot!!

We have seen that some problems composed of dielectric slabs can be described in just one dimension. In this case, the materials and the fields are uniform in two directions. Derivatives in these uniform directions will be zero. We will define the uniform directions to be the x and y axes.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$



$$\begin{aligned} \cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} &= -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}} &= -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} \\ \cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}} &= -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t} \end{aligned}$$



$$\begin{aligned} \cancel{\frac{E_z^{i,j,k+1}|_t - E_z^{i,j,k}|_t}{\Delta y}} - \frac{E_y^{i,j,k+1}|_t - E_y^{i,j,k}|_t}{\Delta z} &= -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \\ \frac{E_x^{i,j,k+1}|_t - E_x^{i,j,k}|_t}{\Delta z} - \cancel{\frac{E_z^{i+1,j,k}|_t - E_z^{i,j,k}|_t}{\Delta x}} &= -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \\ \cancel{\frac{E_y^{i+1,j,k}|_t - E_y^{i,j,k}|_t}{\Delta x}} - \cancel{\frac{E_x^{i,j,k+1}|_t - E_x^{i,j,k}|_t}{\Delta y}} &= -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \end{aligned}$$

$$\begin{aligned} \cancel{\frac{\partial \tilde{H}_z}{\partial y}} - \frac{\partial \tilde{H}_y}{\partial z} &= \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \\ \frac{\partial \tilde{H}_x}{\partial z} - \cancel{\frac{\partial \tilde{H}_z}{\partial x}} &= \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \\ \cancel{\frac{\partial \tilde{H}_y}{\partial x}} - \cancel{\frac{\partial \tilde{H}_x}{\partial y}} &= \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t} \end{aligned}$$

$$\begin{aligned} \cancel{\frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta y}} - \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k}|_{t+\Delta t} - E_x^{i,j,k}|_t}{\Delta t} \\ \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} - \cancel{\frac{\tilde{H}_z^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k}|_{t+\frac{\Delta t}{2}}}{\Delta x}} &= \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k}|_{t+\Delta t} - E_y^{i,j,k}|_t}{\Delta t} \\ \cancel{\frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k}|_{t+\frac{\Delta t}{2}}}{\Delta x}} - \cancel{\frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta y}} &= \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k}|_{t+\Delta t} - E_z^{i,j,k}|_t}{\Delta t} \end{aligned}$$

Decoupling into 2 independent modes

We see that the longitudinal field components E_z and H_z are always zero.

We also see that Maxwell's equations have decoupled into two sets of two equations.

$$-\frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t}$$

$$0 = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}$$

→

Ex/Hy Mode

$$-\frac{E_y^{i,j,k+1}|_t - E_y^{i,j,k}|_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}|_t - E_x^{i,j,k}|_t}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\tilde{H}_z^{i,j,k} = 0$$

Ey/Hx Mode

$$-\frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k}|_{t+\Delta t} - E_x^{i,j,k}|_t}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k}|_{t+\Delta t} - E_y^{i,j,k}|_t}{\Delta t}$$

$$E_z^{i,j,k} = 0$$

2 remaining modes are the same

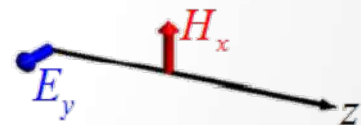
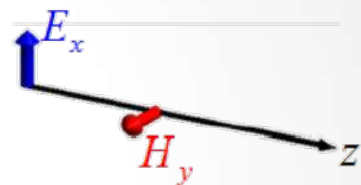
We see that the longitudinal field components E_z and H_z are always zero.

We also see that Maxwell's equations have decoupled into two sets of two equations.

$$\begin{aligned}
 \frac{\partial E_x}{\partial z} &= -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} & \frac{E_x^{i,j,k+1}|_t - E_x^{i,j,k}|_t}{\Delta z} &= -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t} \\
 -\frac{\partial \tilde{H}_y}{\partial z} &= \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} & -\frac{\tilde{H}_y^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k}|_{t+\Delta t} - E_x^{i,j,k}|_t}{\Delta t}
 \end{aligned}$$

➔

$$\begin{aligned}
 \frac{\partial E_y}{\partial z} &= \frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} & \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k}|_{t+\Delta t} - E_y^{i,j,k}|_t}{\Delta t} \\
 \frac{\partial \tilde{H}_x}{\partial z} &= \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} & \frac{E_y^{i,j,k+1}|_t - E_y^{i,j,k}|_t}{\Delta z} &= \frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k}|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}|_{t-\frac{\Delta t}{2}}}{\Delta t}
 \end{aligned}$$



While these modes are physical and would propagate independently, they are numerically the same and will exhibit the same electromagnetic behavior. Therefore, it is only necessary to solve one. We will proceed with the E_y/H_x mode.

E_x/H_y Mode

$$-\frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^k}{c_0} \frac{E_x^k|_{t+\Delta t} - E_x^k|_t}{\Delta t}$$

$$\frac{E_x^{k+1}|_t - E_x^k|_t}{\Delta z} = -\frac{\mu_{yy}^k}{c_0} \frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

 E_y/H_x Mode

$$-\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} = -\frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k|_{t+\Delta t} - E_y^k|_t}{\Delta t}$$

Start with the finite-difference equation which has E_x in the time-derivative:

$$-\frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^k}{c_0} \frac{E_x^k|_{t+\Delta t} - E_x^k|_t}{\Delta t}$$

Solve this for E_x at the future time value.

$$\frac{\epsilon_{xx}^k}{c_0} \frac{E_x^k|_{t+\Delta t} - E_x^k|_t}{\Delta t} = -\frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_x^k|_{t+\Delta t} - E_x^k|_t = -\frac{c_0 \Delta t}{\epsilon_{xx}^k} \frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$\boxed{E_x^k|_{t+\Delta t} = E_x^k|_t + \left(-\frac{c_0 \Delta t}{\epsilon_{xx}^k} \right) \left(\frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)}$$

Start with the finite-difference equation which has E_y in the time-derivative:

$$\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k|_{t+\Delta t} - E_y^k|_t}{\Delta t}$$

Solve this for E_y at the future time value.

$$\frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k|_{t+\Delta t} - E_y^k|_t}{\Delta t} = \frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_y^k|_{t+\Delta t} - E_y^k|_t = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$E_y^k|_{t+\Delta t} = E_y^k|_t + \left(\frac{c_0 \Delta t}{\epsilon_{yy}^k} \right) \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

Start with the finite-difference equation which has H_x in the time-derivative:

$$\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} = \frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Solve this for H_x at the future time value.

$$\frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k|_{t-\frac{\Delta t}{2}}}{\Delta t} = \frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z}$$

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k|_{t-\frac{\Delta t}{2}} = \frac{c_0 \Delta t}{\mu_{xx}^k} \left(\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} \right)$$

$$\boxed{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k|_{t-\frac{\Delta t}{2}} + \frac{c_0 \Delta t}{\mu_{xx}^k} \left(\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} \right)}$$

Start with the finite-difference equation which has H_y in the time-derivative:

$$\frac{E_x^{k+1}|_t - E_x^k|_t}{\Delta z} = -\frac{\mu_{yy}^k}{c_0} \frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Solve this for H_y at the future time value.

$$-\frac{\mu_{yy}^k}{c_0} \frac{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k|_{t-\frac{\Delta t}{2}}}{\Delta t} = \frac{E_x^{k+1}|_t - E_x^k|_t}{\Delta z}$$

$$\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k|_{t-\frac{\Delta t}{2}} = -\frac{c_0 \Delta t}{\mu_{yy}^k} \frac{E_x^{k+1}|_t - E_x^k|_t}{\Delta z}$$

$$\boxed{\tilde{H}_y^k|_{t+\frac{\Delta t}{2}} = \tilde{H}_y^k|_{t-\frac{\Delta t}{2}} - \frac{c_0 \Delta t}{\mu_{yy}^k} \frac{E_x^{k+1}|_t - E_x^k|_t}{\Delta z}}$$

The update coefficients do not change their value during the simulation. They should be computed only once before the main FDTD loop and not at each iteration inside the loop.

The finite-difference equations in terms of the update coefficients are:

$$\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{E_y^{k+1} \Big|_t - E_y^k \Big|_t}{\Delta z} \right)$$

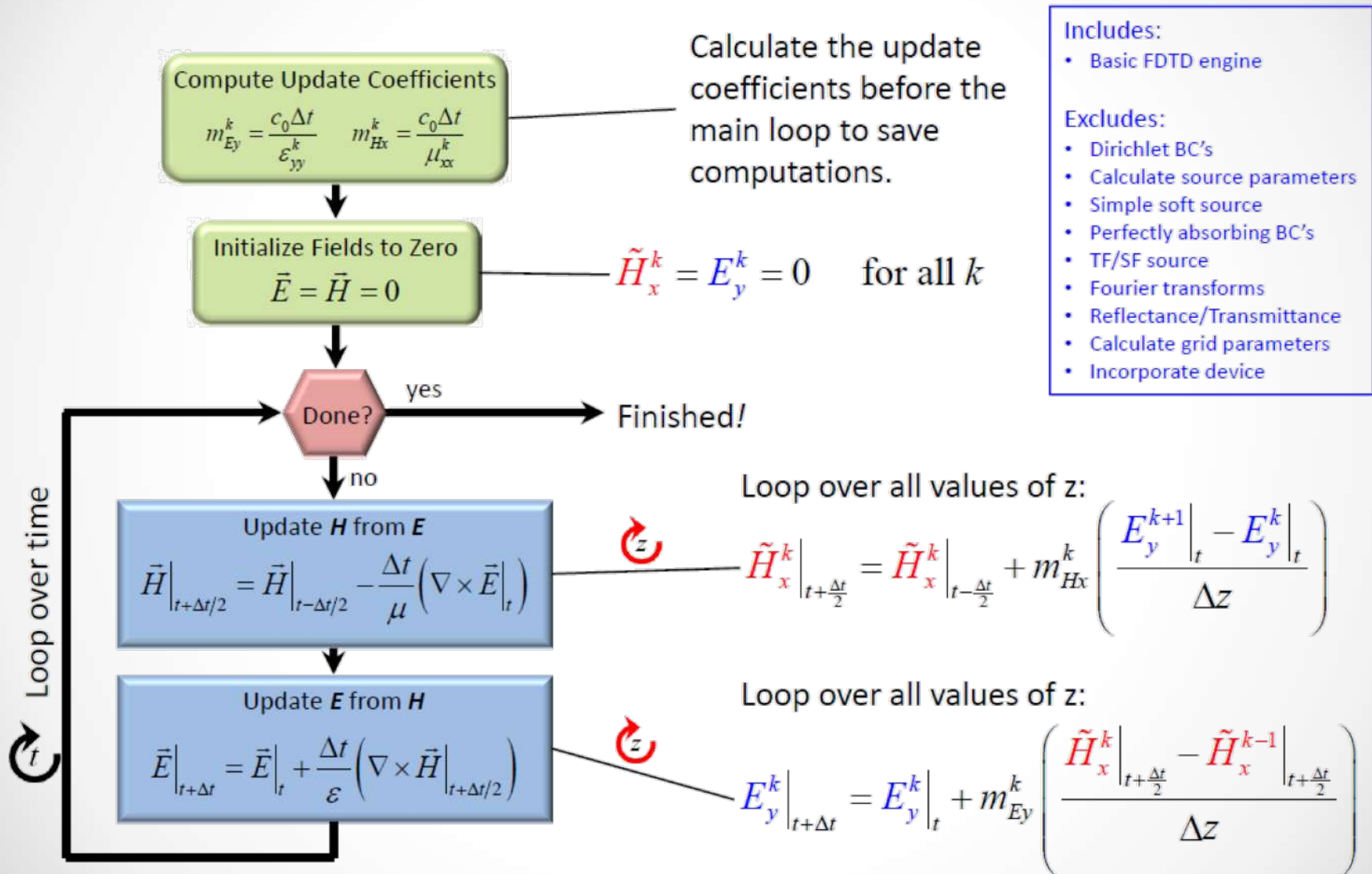
$$E_y^k \Big|_{t+\Delta t} = E_y^k \Big|_t + m_{Ey}^k \left(\frac{\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \Big|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

H_x , E_y , ϵ_{yy} , μ_{xx} , m_{Hx} , and m_{Ey} are all stored in 1D arrays of length N_z .

c_0 , Δt , and Δz are single scalar numbers, not arrays.

$$m_{Ey}^k = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \quad m_{Hx}^k = \frac{c_0 \Delta t}{\mu_{xx}^k}$$

The basic 1D-FDTD algorithm



Update Coefficients

$$m_{Ey}^k = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \quad m_{Hx}^k = \frac{c_0 \Delta t}{\mu_{xx}^k}$$

```
% INITIALIZE MATERIALS TO FREE SPACE
ER = ones(1,Nz);
UR = ones(1,Nz);

% COMPUTE UPDATE COEFFICIENTS
mEy = (c0*dt) ./ ER;
mHx = (c0*dt) ./ UR;
```

Update Equations

You will need to update the fields at every point in the grid so these equations are placed inside a loop from 1 to Nz.

```
% MAIN FDTD LOOP
for T = 1 : STEPS

    % Update H from E
    for nz = 1 : Nz
        Hx(nz) = Hx(nz) + mHx(nz) * (Ey(nz+1) - Ey(nz)) / dz;
    end

    % Update E from H
    for nz = 1 : Nz
        Ey(nz) = Ey(nz) + mEy(nz) * (Hx(nz) - Hx(nz-1)) / dz;
    end

end
```

$$\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{E_y^{k+1} \Big|_t - E_y^k \Big|_t}{\Delta z} \right)$$

$$E_y^k \Big|_{t+\Delta t} = E_y^k \Big|_t + m_{Ey}^k \left(\frac{\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \Big|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

Each cell has its own update equation and its own update coefficients. They are implemented separately for each cell. All of these equations have the same general form so it is more efficient to implement them using a loop. For a 1D grid with 10 cells, think of it this way...

% Update H from E

```
Hx(1) = Hx(1) + mHx(1)*(Ey(2) - Ey(1))/dz;
Hx(2) = Hx(2) + mHx(2)*(Ey(3) - Ey(2))/dz;
Hx(3) = Hx(3) + mHx(3)*(Ey(4) - Ey(3))/dz;
Hx(4) = Hx(4) + mHx(4)*(Ey(5) - Ey(4))/dz;
Hx(5) = Hx(5) + mHx(5)*(Ey(6) - Ey(5))/dz;
Hx(6) = Hx(6) + mHx(6)*(Ey(7) - Ey(6))/dz;
Hx(7) = Hx(7) + mHx(7)*(Ey(8) - Ey(7))/dz;
Hx(8) = Hx(8) + mHx(8)*(Ey(9) - Ey(8))/dz;
Hx(9) = Hx(9) + mHx(9)*(Ey(10) - Ey(9))/dz;
Hx(10) = Hx(10) + mHx(10)*(Ey(11) - Ey(10))/dz;
```

% Update H from E

```
for nz = 1 : 10
    Hx(nz) = Hx(nz) + mHx(nz)*(Ey(nz+1) - Ey(nz))/dz;
end
```

% Update E from H

```
Ey(1) = Ey(1) + mEy(1)*(Hx(1) - Hx(0))/dz;
Ey(2) = Ey(2) + mEy(2)*(Hx(2) - Hx(1))/dz;
Ey(3) = Ey(3) + mEy(3)*(Hx(3) - Hx(2))/dz;
Ey(4) = Ey(4) + mEy(4)*(Hx(4) - Hx(3))/dz;
Ey(5) = Ey(5) + mEy(5)*(Hx(5) - Hx(4))/dz;
Ey(6) = Ey(6) + mEy(6)*(Hx(6) - Hx(5))/dz;
Ey(7) = Ey(7) + mEy(7)*(Hx(7) - Hx(6))/dz;
Ey(8) = Ey(8) + mEy(8)*(Hx(8) - Hx(7))/dz;
Ey(9) = Ey(9) + mEy(9)*(Hx(9) - Hx(8))/dz;
Ey(10) = Ey(10) + mEy(10)*(Hx(10) - Hx(9))/dz;
```

% Update E from H

```
for nz = 1 : 10
    Ey(nz) = Ey(nz) + mEy(nz)*(Hx(nz) - Hx(nz-1))/dz;
end
```

Must resolve the minimum wavelength

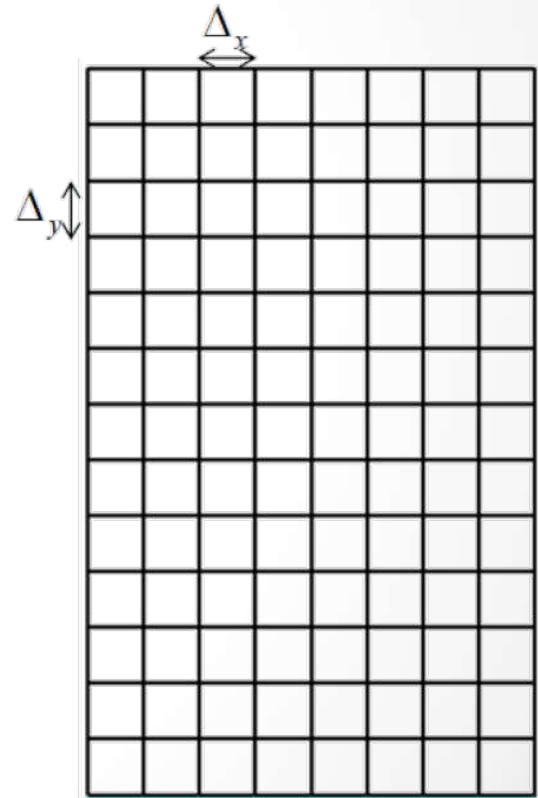
$$\Delta_\lambda = \frac{\lambda_{\min}}{N_\lambda} \quad N_\lambda \geq 10$$

Must resolve the minimum structural dimension

$$\Delta_d = \frac{d_{\min}}{N_d} \quad N_d \geq 1$$

Set the initial grid resolution to the smallest number computed above

$$\Delta_x = \Delta_y = \min[\Delta_\lambda, \Delta_d]$$



Decide what dimensions along each axis are critical

→ Typically this is a lattice constant or grating period along x

→ Typically this is a layer thickness along y

$$d_x \text{ and } d_y$$

Compute how many grid cells comprise d_c , and round UP

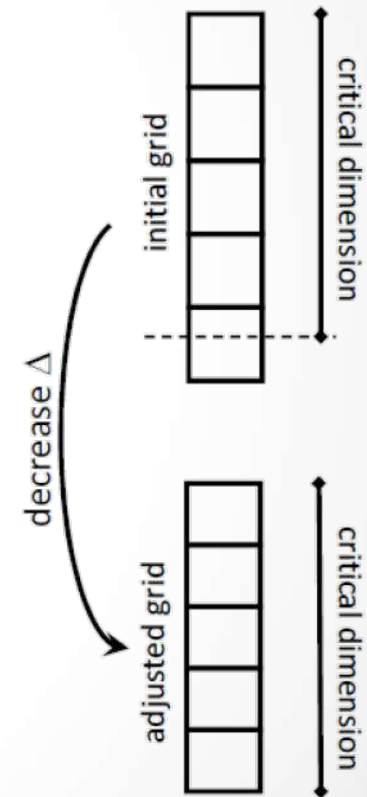
$$M_x = \text{ceil}(d_x / \Delta_x)$$

$$M_y = \text{ceil}(d_y / \Delta_y)$$

Adjust grid resolution to fit this dimension in grid EXACTLY

$$\Delta_x = d_x / M_x$$

$$\Delta_y = d_y / M_y$$



Over a time duration of one time step Δt , an electromagnetic disturbance will travel:

Numerical distance covered in one time step: Δz

Physical distance covered in one time step: $c_0 \Delta t / n$

Because of the numerical algorithm, it is not possible for a disturbance to travel farther than one unit cell in a single time step.

We need to make sure that a physical wave would not propagate farther than a single unit cell in one time step.

$$\frac{c_0 \Delta t}{n} < \Delta z$$

This places an upper limit on the time step.

$$\Delta t < \frac{n \Delta z}{c_0}$$

n should be set to the smallest refractive index found anywhere in the grid. Usually this is just made to be 1.0 and dropped from the equation.

Define the time-step

Refractive index is greater than or equal to one, so our condition on Δt can be written more simply as:

$$\Delta t < \frac{\Delta z}{c_0} \quad \leftarrow \text{Sort of a worst case.}$$

$n=1$ produces the fastest possible physical wave.

For 2D or 3D grids, the condition can be generalized as

The Courant stability condition

$$\Delta t < \frac{1}{c_0 \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}$$

The stability condition will be most restrictive along the shortest dimension of the grid unit cell.

$$\Delta_{\min} = \min[\Delta x, \Delta y, \Delta z]$$

A good equation to ensure stability and accuracy on any grid is then

$$\Delta t < \frac{\Delta_{\min}}{2c_0}$$

Note: A factor of 0.5 was included here as a safety margin.

This can be generalized to account for special cases.

$$\Delta t < \frac{n_{\min} \Delta_{\min}}{2c_0}$$

1. Your grid is filled with dielectric and travels slower everywhere.
2. Your model includes dispersive materials with refractive index less than one.

We must implement the update equations for every point in the grid.

$$\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{E_y^{k+1} \Big|_t - E_y^k \Big|_t}{\Delta z} \right)$$

What do we do at $k = N_z$?

$E_y^{N_z+1}$ does not exist.

$$E_y^k \Big|_{t+\Delta t} = E_y^k \Big|_t + m_{Ey}^k \left(\frac{\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \Big|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

What do we do at $k = 1$?

$\tilde{H}_x^0$ does not exist.

Conditions

- Waves at the boundaries are only travelling outward.
- Materials at the boundaries are linear, homogeneous, isotropic and non-dispersive
- The refractive index is the same at both boundaries and is n_{bc} .
- Time step is chosen so physical waves travel 1 cell in exactly two time steps

$$\Delta t = n_{bc} \Delta z / (2c_0)$$

Implementation at z-Low Boundary

At the z-low boundary, we need only modify the E-field update equation.

$$h_3 = h_2 \quad h_2 = h_1 \quad h_1 = \tilde{H}_x^1 \quad E_y^1|_{t+\Delta t} = E_y^1|_t + m_{Ey}^k \left(\frac{\tilde{H}_x^1|_{t+\frac{\Delta t}{2}} - h_3}{\Delta z} \right)$$

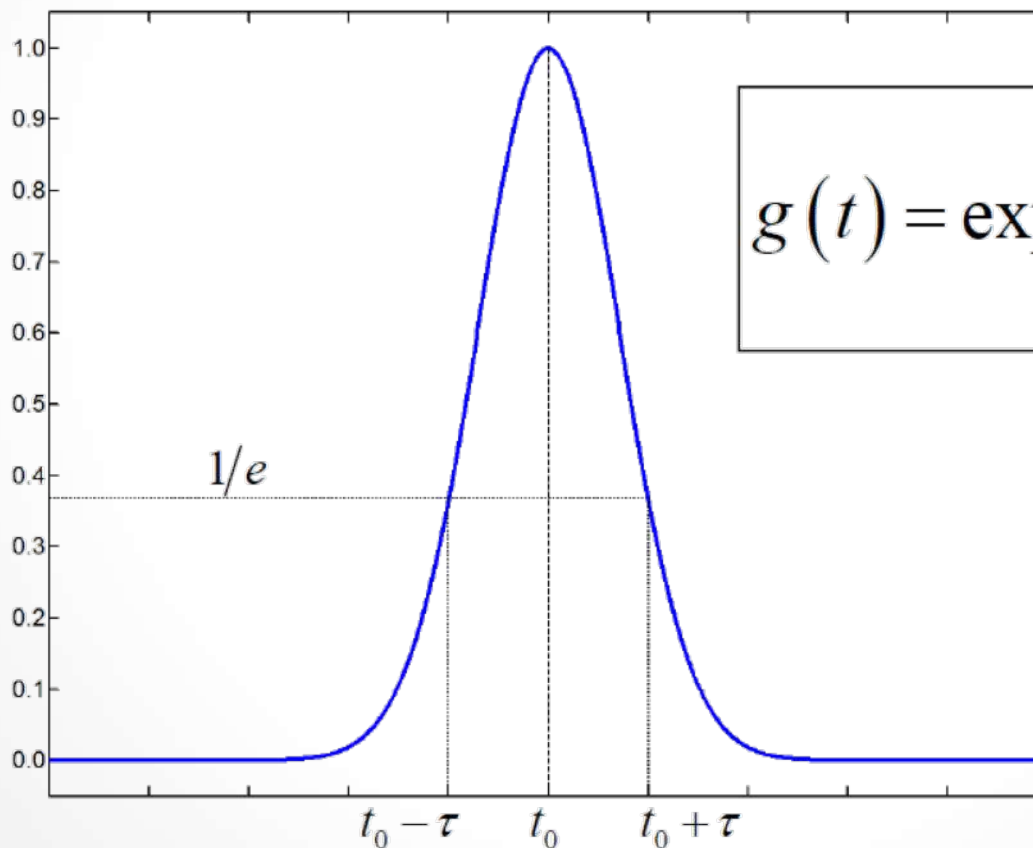
Implementation at z-High Boundary

At the z-high boundary, we need only modify the H-field update equation.

$$e_3 = e_2 \quad e_2 = e_1 \quad e_1 = E_y^{N_z} \quad \tilde{H}_x^{N_z}|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^{N_z}|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{e_3 - E_y^{N_z}|_t}{\Delta z} \right)$$

The Gaussian pulse

In FDTD, we typically use short pulses as sources. This approximates an impulse function so we can excite a single simulation with a broad range of frequencies at the same time.



$$g(t) = \exp\left[-\left(\frac{t-t_0}{\tau}\right)^2\right]$$

τ is the half width at $1/e$

The Gaussian pulse

The pulse source must start at zero and gradually increase. NO STEP FUNCTIONS!!

$$t_0 \geq 6\tau$$



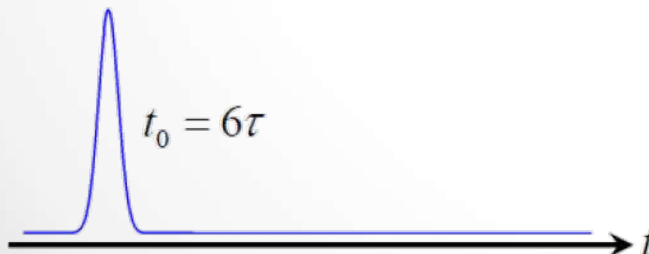
WRONG!!

The step function at the beginning will induce very large field gradients.



STILL WRONG!!

While better, this source still starts with a step function that will produce large field gradients.



CORRECT!!

This source “eases” into the Gaussian source.

Source #1: Simple Hard Source

The simple hard source is the easiest to implement. After updating the field across the entire grid, one field component at one point on the grid is overwritten with the source. This approach injects power into the model, but the source point behaves like a perfect electric conductor or perfect magnetic conductor and will scatter waves.

$$\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} = g_H \Big|_k \quad \text{and/or} \quad E_y^k \Big|_{t+\Delta t} = g_E \Big|_k$$

OVERWRITE

Not usually a practical source.
We won't use it.

Source #2: Simple Soft Source

The simple soft source is better than the hard source because it is transparent to scattered waves passing through it. After updating the field across the entire grid, the source function is added to one field component at one point on the grid. This approach injects power into the model in both directions. It is great for testing boundary conditions.

$$\tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \Big|_{t+\frac{\Delta t}{2}} + g_H \Big|_k \quad \text{and/or} \quad E_y^k \Big|_{t+\Delta t} = E_y^k \Big|_{t+\Delta t} + g_E \Big|_k$$