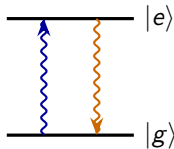


Photon–Atom Coupling

From the Dipole Interaction to Cavity QED and the Optical Response of Solids

Photonics Engineering — Lecture course



How to read this course: the colour code

Why do we need a new theory? (teal)

What breaks, and why you should care, before any algebra starts.

Key result (blue)

The few results to remember; the skeleton of the course.

Demonstration (orange)

Derivations worked line by line; the exam asks you to reproduce one.

Application (green)

Real numbers for real systems; redo them with a calculator.

Physical insight (grey)

What a result means; the picture to keep in your head.

Why it matters (violet)

The sky, MRI, GPS, quantum processors: why you are learning this.

Check yourself (yellow)

End-of-chapter questions; answer them out loud, without notes.

Deliberate ordering: two-level atom (L1) $\rightarrow$ coherent dynamics (L2) $\rightarrow$ dissipation (L3), so the density matrix does not land on you all at once.

Prerequisites, notation, learning outcomes

Prerequisites. Two-level quantum systems, time-dependent perturbation theory, Maxwell equations and Poynting flux, harmonic oscillator, density of states. SI units; orders of magnitude for real systems throughout.

Notation. $|g\rangle, |e\rangle$; $\omega_0 = (E_e - E_g)/\hbar$; field ω ; detuning $\delta = \omega - \omega_0$; dipole $\mathbf{d} = \langle e | \hat{\mathbf{d}} | g \rangle$; Rabi Ω ; decay Γ .

Assessment. Problem set per block (11 worked solutions at the end) + exam: one derivation (orange) and one estimate (green).

By the end you should be able to:

- ▶ write $-\mathbf{d} \cdot \mathbf{E}$, justify dipole & RWA (L1–2)
- ▶ solve coherent dynamics: Rabi, Bloch sphere, π -pulses, Ramsey, light shift (L2)
- ▶ add dissipation: optical Bloch, saturation, power broadening, echo (L3)
- ▶ derive Fermi's golden rule and apply it (L4, L8)
- ▶ explain why spontaneous emission needs a quantised field; compute Γ , linewidths (L5)
- ▶ connect dipole to susceptibility; absorption, dispersion, slow light (L6)
- ▶ weak vs strong coupling; Purcell factor; size a single-photon source (L7)
- ▶ build $\varepsilon(\omega)$ of a solid; Kramers–Kronig; bands, excitons, polaritons (L8)

0.1–0.2 Two ingredients, one question; the quantum turn

Two characters: the electromagnetic field and electric charges. Maxwell: light is a travelling EM wave. The organising question: **by what mechanism, and at what rate, do they exchange energy?**

Three experiments broke the classical picture

(i) blackbody spectrum (energy diverges at high ω); (ii) photoelectric effect (packets $\hbar\omega$); (iii) sharp spectral lines (element fingerprints).

The quantum turn

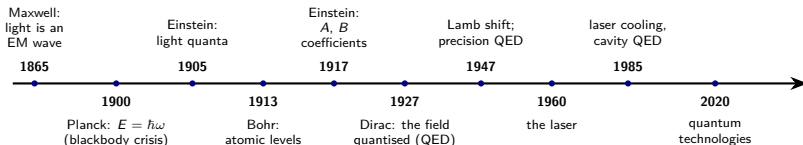
Both partners are *lumpy*: discrete levels (Bohr) and photons (Planck, Einstein). A spectral line = a photon matching a level spacing.

The subtlety that drives the course

Semiclassical: quantise the atom, keep the field classical. Excellent for *driven* processes. But an excited atom *in the dark* still decays: nothing classical can cause that. Spontaneous emission forces field quantisation (L5).

0.3 Crisis as method: how physics actually advances

Physics advances when an excellent theory is pushed into a corner where it *visibly fails*, and the failure is taken seriously. Planck (1900): packets $\hbar\omega$, “an act of desperation”. Einstein (1905) took them literally; Bohr (1913) quantised the atom with rules he could not justify; the scaffolding held until quantum mechanics (1925–26) was built under it.



The general fate of good theories

An old theory that worked never becomes wrong; it becomes a **limiting case**. Watch it happen three times here: Lorentz inside the quantum χ (L6); rate equations as the strong-damping limit of Bloch (L3–4); the semiclassical theory inside QED.

0.4 Models, not mirrors

- ▶ The “two-level atom” does not exist; the “electron on a spring” even less. Both are spectacularly predictive *within stated limits*.
- ▶ A model is judged by **the questions it answers correctly**; its domain of validity is part of the model.
- ▶ Truncating sodium to two levels claims only: near one line, nature *answers as if*.

Even the photon deserves scepticism

No position operator; delocalised over the mode; Lamb's 1995 polemic (“Anti-photon”). What is beyond dispute is the *structure*: quantised exchanges. Vacuum fluctuations “really there”? Nature answers with measured numbers: Lamb shift, Casimir force, spontaneous emission. We use the picture that makes the calculation transparent; experiment arbitrates.

0.5–0.6 Quantising the field; one elementary process

Each mode $\rightarrow$ a quantum harmonic oscillator;
excitations = photons. The vacuum is **not empty**:
zero-point fluctuations knock the excited atom down
(spontaneous emission) and shift levels (Lamb shift).
The full theory: **QED**.

One vertex

A charge emits or absorbs one photon; everything else is assemblies of this event. Strength: $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$. Small $\Rightarrow$ perturbation theory works (L4).



Left to right: the fundamental vertex; absorption; emission; self-energy loop (the origin of the Lamb shift). Straight lines: electrons; wavy: photons.

0.7–0.8 Where we stand today; what is assumed

QED: the most accurately tested theory (electron magnetic moment: ~ 12 significant figures); template of the Standard Model. Yet at optical frequencies we almost never need the full apparatus: $-\mathbf{d} \cdot \mathbf{E}$ is the low-energy face of the vertex.

More is different (Anderson)

A solid is 10^{23} atoms obeying the same coupling, yet bands, excitons, plasma edges, polaritons are *emergent*: derivable in hindsight, unforeseeable in advance. This course walks the ladder both ways.

Self-contained toolbox (appendices)

Appendix C: Schrödinger equation, stationary states.

Appendix D: perturbation theory.

Appendix E: harmonic oscillator, second quantisation.

Appendix A: rotating frame, master equation, mode counting.

Appendix B: constants. Appendix F: formula sheet.

This chapter is the map; the rest of the course is the territory.

1. Why start with a classical oscillator?

Free intuition before the quantum price

A charge on a spring, pushed by a light wave, already reproduces **absorption**, **refraction** and the coloured sky, and hands us the right orders of magnitude. We keep it until it visibly fails (no saturation, no discrete levels); that failure forces the two-level atom at the end of this chapter.

Roadmap, and the symbols introduced here

(you are here) classical oscillator $\rightarrow$ two-level atom. New symbols: ω_0 , γ_c (classical damping), $\alpha(\omega)$, $\chi(\omega)$, $\mathbf{d}$, $H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}$.

Lorentz model: electron bound to the nucleus, driven by $\mathbf{E}(t) = \mathbf{E}_0 \cos \omega t$:

$$m_e \ddot{\mathbf{r}} + m_e \gamma_c \dot{\mathbf{r}} + m_e \omega_0^2 \mathbf{r} = -e \mathbf{E}_0 \cos \omega t$$

1.1 Solving the driven oscillator

Solving the driven oscillator

Seek $\mathbf{r}(t) = \text{Re}[\mathbf{r}_0 e^{-i\omega t}]$: the differential equation becomes algebraic,

$$(-\omega^2 - i\gamma_c\omega + \omega_0^2)\mathbf{r}_0 = -\frac{e}{m_e}\mathbf{E}_0 \Rightarrow \mathbf{r}_0 = \frac{-e/m_e}{\omega_0^2 - \omega^2 - i\gamma_c\omega}\mathbf{E}_0.$$

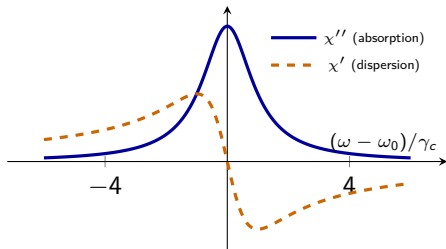
Dipole $\mathbf{p}_0 = -e\mathbf{r}_0 = \alpha(\omega)\mathbf{E}_0$; polarization $\mathbf{P} = N\mathbf{p} = \epsilon_0\chi\mathbf{E}$. The complex denominator = the 90° phase lag at resonance = absorption.

Lorentz polarizability and susceptibility

$$\alpha(\omega) = \frac{e^2/m_e}{\omega_0^2 - \omega^2 - i\gamma_c\omega}, \quad \chi(\omega) = \frac{Ne^2/(m_e\epsilon_0)}{\omega_0^2 - \omega^2 - i\gamma_c\omega}$$

$\text{Re } \chi$: dispersion (index); $\text{Im } \chi$: absorption, Lorentzian of width γ_c near resonance.

1.1 The universal resonant lineshape



Worth memorising

One resonant denominator, real (dispersive) + imaginary (absorptive) parts. Absorption peaks where dispersion crosses zero; steepest dispersion at line centre. It returns in L6 (quantum χ) and L8 (solids), tied by Kramers–Kronig.

1.1 Scattering that grows as ω^4 : the colour of the sky

Why the sky is blue

(i) An oscillating dipole is an accelerating charge: $P_{\text{scatt}} \propto (\ddot{p})^2 \propto \omega^4 p_0^2$. (ii) Far below resonance ($\omega \ll \omega_0$: air in the visible, resonances in the UV), $\alpha \approx e^2/m_e\omega_0^2$ is *constant*: p_0 is the same for every colour.

$$\frac{P_{\text{blue}}}{P_{\text{red}}} = \left(\frac{\lambda_{\text{red}}}{\lambda_{\text{blue}}} \right)^4 \approx \left(\frac{650}{450} \right)^4 \approx 4.4$$

Blue is scattered out of the sunbeam a few times more than red: blue sky, red low sun.

The real part does the complementary job

χ' fixes $n \simeq 1 + \chi'/2$; its slow growth with ω bends blue more than red: dispersion, prisms, chromatic aberration. Absorption and refraction are the two halves of one denominator.

1.2 The atom–field Hamiltonian: from minimal coupling

Classical electrodynamics hands us the Hamiltonian (minimal coupling $\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A}$, the unique prescription reproducing the Lorentz force):

$$H = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A})^2 - e\phi + V(\mathbf{r})$$

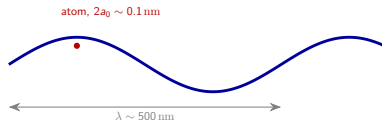
From minimal coupling to $\mathbf{A} \cdot \mathbf{p}$

Step 1. Expand; in Coulomb gauge $\mathbf{p}$ and $\mathbf{A}$ commute:

$$H_{\text{int}} = \frac{e}{m_e}\mathbf{A} \cdot \mathbf{p} + \frac{e^2}{2m_e}A^2$$

Step 2. Ratio of the two terms $\sim eA_0/2p \sim eE_0/2\omega m_e \alpha c \sim 10^{-9}$ at $E_0 \sim 275 \text{ V/m}$: drop A^2 (it matters only at 10^{14} – 10^{15} W/cm^2 : high-harmonic generation, strong-field optics).

1.2 Dipole approximation, and the length gauge



$ka_0 \sim 10^{-3}$: $e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1$; the atom feels $\mathbf{E}(t)$, not $\mathbf{E}(\mathbf{r}, t)$.

When does it fail?

(i) $\lambda \sim$ system size (X-rays, Rydberg atoms); (ii) $\mathbf{d}_{fi} = 0$ by symmetry: magnetic-dipole / quadrupole terms of order ka_0 take over: weak “forbidden” lines.

From $\mathbf{A} \cdot \mathbf{p}$ to $-\mathbf{d} \cdot \mathbf{E}$

Step 1. $[\mathbf{r}, H_0] = i\hbar\mathbf{p}/m_e$ between eigenstates: $\langle f | \mathbf{p} | i \rangle = im_e\omega_{fi} \langle f | \mathbf{r} | i \rangle$ (the quantum $p = m\dot{x}$).

Step 2. With $E_0 = \omega A_0$ and $\omega \simeq \omega_{fi}$:

$$\left| \frac{e}{m_e} A_0 \langle f | \mathbf{p} | i \rangle \right| \simeq eE_0 |\langle f | \mathbf{r} | i \rangle|$$

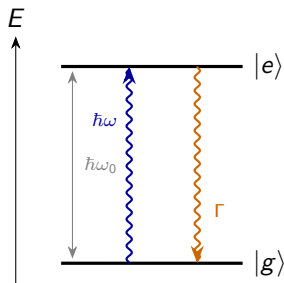
identical matrix elements on resonance. (Systematic route: Power–Zienau–Woolley transformation.)

Electric-dipole interaction

$$H_{\text{int}} = -\hat{\mathbf{d}} \cdot \mathbf{E}(\mathbf{r}_0, t), \quad \hat{\mathbf{d}} = -e\hat{\mathbf{r}}$$

The workhorse of the whole course.

1.2 The two-level atom, pictured



The three Einstein processes:

- ▶ **absorption** of $\hbar\omega$ (scales with the field)
- ▶ **stimulated emission** (scales with the field)
- ▶ **spontaneous emission** at rate Γ (requires the quantised vacuum, L5)

$\mathbf{d} = \langle e | \hat{\mathbf{d}} | g \rangle$ sets the strength of every light-matter process in this course.

1.3 Orders of magnitude: sodium

Dipole moment and Rabi scale of Na ($D_2, 3s \rightarrow 3p$)

$\lambda_0 = 589 \text{ nm}$, $\omega_0 = 3.2 \times 10^{15} \text{ rad/s}$; $d \simeq 2.5 ea_0 \approx 2.1 \times 10^{-29} \text{ C m}$ (6.3 D).

Intensity $\rightarrow$ field: $I = \frac{1}{2} \epsilon_0 c E_0^2$ (Poynting, $\langle \cos^2 \rangle = \frac{1}{2}$), so at $I = 10 \text{ mW/cm}^2$:

$$E_0 = \sqrt{2I/\epsilon_0 c} \approx 275 \text{ V/m} \quad \Rightarrow \quad \Omega = \frac{dE_0}{\hbar} \approx 5.5 \times 10^7 \text{ rad/s}, \quad \Omega/2\pi \approx 8.7 \text{ MHz}$$

Calibration: sunlight $E_0 \approx 870 \text{ V/m}$; 1 mW in $1 \mu\text{m}$: $7 \times 10^5 \text{ V/m}$; internal atomic field $\sim 5 \times 10^{11} \text{ V/m} \Rightarrow$ lasers are weak perturbations.

The race that organises the course

Control time $t_\pi = \pi/\Omega \approx 60 \text{ ns}$ vs decay time $\tau \approx 16 \text{ ns}$. $t_\pi \ll \tau$: clean oscillations. $t_\pi \gtrsim \tau$ (here): decay wins, rate picture. We cannot make this comparison properly yet: that is why L2 and L3 exist.

1.4 From the Lorentz oscillator to the two-level atom

Fatal flaws of the classical model: response *linear* at any strength, spectrum *continuous*. A real atom saturates (proved in L3) and has discrete levels. Near resonance with one transition, keep two levels:

The bare two-level atom

$$H_0 = \hbar\omega_0 |e\rangle\langle e|, \quad \omega_0 = \frac{E_e - E_g}{\hbar}$$

ω_0 survives from Lorentz; the response is now bounded: at most one quantum stored.

The thread: one chain, six demonstrations

1. **Rabi** (Schrödinger, 2×2) 2. **Bloch** ($+T_1, T_2$: line shape, saturation) 3. **Golden rule** ($\text{sinc}^2 \rightarrow \delta$: a rate) 4. **Spontaneous emission** (golden rule on the vacuum) 5. **Purcell** (recount the modes) 6. **Solids** (bands, excitons). Reproduce these six and you own the subject.

The hub of the course

The same two-level object returns: driven, damped, in vacuum, in a cavity, replicated 10^{23} times into bands.

Check yourself (Chapter 1)

Can you answer these out loud?

- ▶ Why does the classical oscillator scatter blue light much more than red?
- ▶ What do the real and imaginary parts of $\chi(\omega)$ each describe physically?
- ▶ Why is the dipole approximation ($e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1$) so good at optical wavelengths?
- ▶ In one line: why must we eventually abandon the Lorentz oscillator for a two-level atom?

2. Why is a rate equation not enough?

Rates cannot describe control

Einstein's picture assigns a probability per second to each jump: fine for a light bulb. It cannot describe an atom driven coherently *back and forth*, a superposition, or the phase that makes paths interfere: precisely a laser pulse on an atom, and every quantum bit. We must track the *amplitudes* c_g, c_e with the full Schrödinger equation.

Roadmap, and the symbols introduced here

✓ oscillator ✓ two-level atom → **(you are here)** coherent dynamics. New: $\Omega = dE_0/\hbar$, $\delta = \omega - \omega_0$, $\Omega_R = \sqrt{\Omega^2 + \delta^2}$, Bloch vector (u, v, w) , U_{dip} .

This chapter is the working principle of trapped-ion and superconducting quantum computers, of the atomic clocks inside GPS, and of optical tweezers. Relaxation, their common enemy, is Chapter 3.

2.1 Rabi oscillations and the RWA (i): to the rotating frame

Rotating-wave approximation, step by step

Step 1: coupled amplitudes. With $H_{\text{int}} = -dE_0 \cos \omega t (|e\rangle\langle g| + |g\rangle\langle e|)$ and the free phase factored out,

$$i\dot{c}_e = -\frac{dE_0}{2\hbar} \left(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t} \right) c_g$$

Step 2: the RWA. First exponential: slow (at δ). Second: at $\omega + \omega_0 \sim 2\omega_0$, averages to zero over an optical cycle: drop it.

Why the next move? An explicit $e^{i\delta t}$ remains: go to the frame co-rotating with the laser (the carousel picture); the Hamiltonian becomes time-independent:

$$\tilde{H} = -\hbar\delta |e\rangle\langle e| + \frac{\hbar\Omega}{2} (|e\rangle\langle g| + |g\rangle\langle e|), \quad \Omega = \frac{dE_0}{\hbar}$$

Step 3: diagonalise. $\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega \\ \Omega & -2\delta \end{pmatrix}$: eigenvalue splitting $\Omega_R = \sqrt{\Omega^2 + \delta^2}$.

2.1 Rabi oscillations (ii): solving the dynamics, in full

Step 4: time evolution, every line

Equations: $i\dot{c}_g = \frac{\Omega}{2}c_e$, $i\dot{c}_e = \frac{\Omega}{2}c_g - \delta c_e$. **On resonance** ($\delta = 0$): $\ddot{c}_e = -(\Omega/2)^2 c_e$, harmonic; with $c_e(0) = 0$, $\dot{c}_e(0) = -i\Omega/2$: $c_e = -i \sin(\Omega t/2)$, so $P_e = \sin^2(\Omega t/2)$.

Off resonance: eliminate c_g : $\ddot{c}_e - i\delta\dot{c}_e + (\Omega/2)^2 c_e = 0$. Try $c_e \propto e^{i\lambda t}$: $\lambda_{\pm} = \frac{1}{2}(\delta \pm \Omega_R)$. Then $c_e = e^{i\delta t/2}(Ae^{i\Omega_R t/2} + Be^{-i\Omega_R t/2})$; $c_e(0) = 0 \Rightarrow B = -A$; $\dot{c}_e(0) = -i\Omega/2 \Rightarrow A = -\Omega/2\Omega_R$:

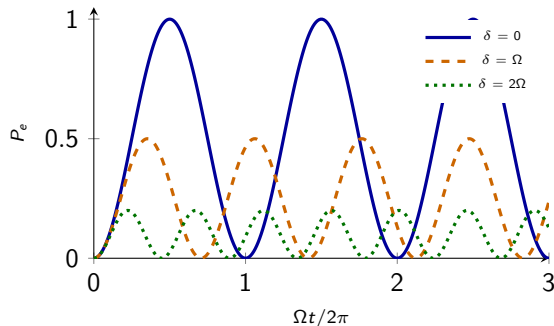
$$c_e(t) = -i \frac{\Omega}{\Omega_R} e^{i\delta t/2} \sin \frac{\Omega_R t}{2}$$

Generalized Rabi oscillations

$$P_e(t) = \frac{\Omega^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R t}{2}\right), \quad \Omega_R = \sqrt{\Omega^2 + \delta^2}$$

π -pulse inverts; $\pi/2$ -pulse makes the equal superposition. No rate equation can reproduce this cycling.

2.1 Rabi oscillations (iii): the picture, and the gate

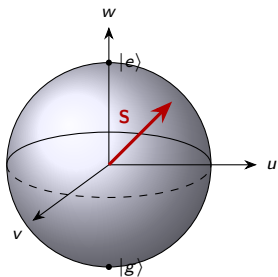


A Rabi pulse is a quantum logic gate

$|g\rangle, |e\rangle = \text{"0"}, \text{"1"}$. π -pulse = NOT gate; $\pi/2$ -pulse creates the superposition that powers quantum computing. Trapped-ion and superconducting processors do nothing else at hardware level: timed Rabi pulses, finished before relaxation scrambles the phase. This equation is their instruction set.

Detuned: faster (Ω_R) but shallower (Ω^2/Ω_R^2): a rotation about a tilted axis never reaches the pole.

2.2 The Bloch sphere: coherent rotations



$$u = 2 \operatorname{Re} c_e c_g^*, \quad v = 2 \operatorname{Im} c_e c_g^*, \quad w = |c_e|^2 - |c_g|^2.$$

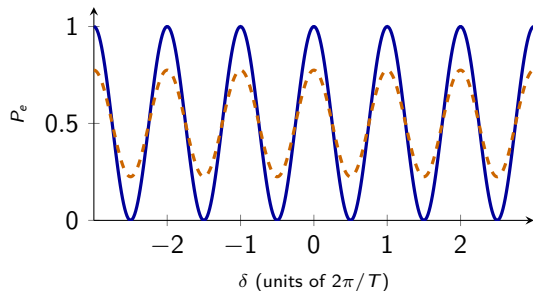
Why three numbers, why a unit sphere? (c_g, c_e): 4 real parameters – normalisation – global phase = 2 angles. And

$$u^2 + v^2 + w^2 = (|c_e|^2 + |c_g|^2)^2 = 1$$

Pure states live *on* the sphere; the interior (mixed states) opens up with the ensemble averaging of Chapter 3.

A resonant pulse of area $\theta = \Omega t$ rotates $\mathbf{S}$ by θ about an equatorial axis set by the laser phase: π -pulse south $\rightarrow$ north; $\pi/2$ -pulse lands on the equator.

2.2 Ramsey interferometry, and why GPS works



Two $\pi/2$ -pulses separated by T : $P_e = \cos^2(\delta T/2)$ (Problem 7). Fringe spacing $2\pi/T$; decoherence damps the contrast as e^{-T/T_2} (dashed).

This is how GPS knows where you are

The second is *defined* by a Ramsey sequence. Longer $T \Rightarrow$ finer fringes; optical clocks reach 10^{-18} (a second over the age of the Universe). GPS positions come from ns-level clock differences: a few ns = a metre on the ground.

2.3 Dressed states and the light shift

Light shift from the dressed-state energies

Step 1: eigenvalues. $\lambda^2 + \hbar\delta\lambda - (\hbar\Omega/2)^2 = 0$:

$$E_{\pm} = -\frac{\hbar\delta}{2} \pm \frac{\hbar}{2}\sqrt{\Omega^2 + \delta^2}$$

Step 2: expand for $|\delta| \gg \Omega$: $\sqrt{\Omega^2 + \delta^2} \simeq |\delta| + \Omega^2/2|\delta|$; the branch connected to the bare ground state shifts by $\Delta E_g \simeq +\hbar\Omega^2/4\delta$ (levels *repel*: second-order perturbation theory with denominator $\hbar\delta$).

Step 3: laboratory form. $I/I_{\text{sat}} = 2\Omega^2/\Gamma^2$ (L3) gives $\Delta E_g = \frac{\hbar\Gamma^2}{8\delta} \frac{I}{I_{\text{sat}}}$; eliminating d with $d^2 = 3\pi\epsilon_0\hbar c^3\Gamma/\omega_0^3$ (L5) gives the boxed form.

Step 4: the sign. Red detuning ($\delta < 0$): bright regions attract. Blue: they repel (hollow traps, atom mirrors).

Optical dipole potential

$$U_{\text{dip}}(\mathbf{r}) = \frac{\hbar\Omega^2(\mathbf{r})}{4\delta} = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\delta} I(\mathbf{r})$$

Trap $\propto I/\delta$, scattering $\propto I/\delta^2$: far-off-resonance tweezers.

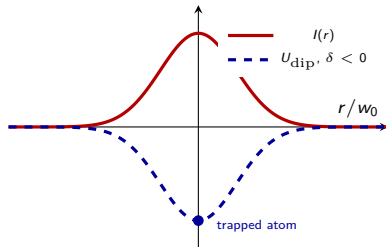
2.3 The optical tweezer, in numbers and in picture

A far-detuned tweezer

1064 nm on Na ($\lambda_0 = 589$ nm): $\delta/2\pi \sim -2\text{e}14$ Hz. 10 mW into $1\text{ }\mu\text{m}$: $I = 2P/\pi w^2 \approx 6\text{e}9$ W/m², depth $U/k_B \sim 0.3$ mK, a few scattered photons/s: holds a $\sim 10\text{ }\mu\text{K}$ laser-cooled atom for seconds.

Tweezers and atom arrays

Ashkin, Nobel 2018; laser cooling, Nobel 1997. Hundreds of single atoms in programmable arrays: a leading quantum-computing platform. All of it is $U \propto I/\delta$ plus suppressed scattering.



The intensity maximum becomes a potential minimum: a marble in a bowl of light.

Check yourself (Chapter 2)

Can you answer these out loud?

- ▶ What is the physical meaning of the Rabi frequency Ω ?
- ▶ Why do we move to a frame rotating at the laser frequency before solving?
- ▶ What does a π -pulse do to the atom, and a $\pi/2$ -pulse?
- ▶ Why can a simple absorption/emission rate never reproduce Rabi flopping?

3. Why is the Schrödinger equation no longer enough?

A real atom is never isolated

It emits photons spontaneously; collides with atoms or phonons; sees a jittering laser; feels stray fields. After a random event we know only the *statistics* of the wavefunction: one $|\psi\rangle$ cannot hold “populations we know” with “coherences being scrambled”. The new object: the density matrix.

Roadmap, and the symbols introduced here

✓ coherent dynamics → **(you are here)** dissipation. New: ρ , T_1 (energy), T_2 (phase), T_2^* (pure dephasing), s (saturation).

Relaxation is both the great enemy (limits every quantum computer and clock) and a tool (laser cooling, MRI contrast). T_1 and T_2 are among the most-quoted numbers in physics and medicine.

3.1 From an ensemble of wavefunctions to the density matrix

The averaging, completely explicit

N identically prepared atoms; atom k : $|\psi_k\rangle = c_g |g\rangle + c_e e^{i\varphi_k} |e\rangle$, own random phase (own collision history).

Step 1: populations survive. $\overline{|c_e|^2} = |c_e|^2$: the φ_k drop out.

Step 2: the coherence does not. $\overline{c_e c_g^*} = c_e c_g^* \overline{e^{i\varphi}}$; N random unit vectors sum to $\sim \sqrt{N}$: $\overline{e^{i\varphi}} \rightarrow 0$. Every atom stays pure; what dies is the *agreement between atoms*.

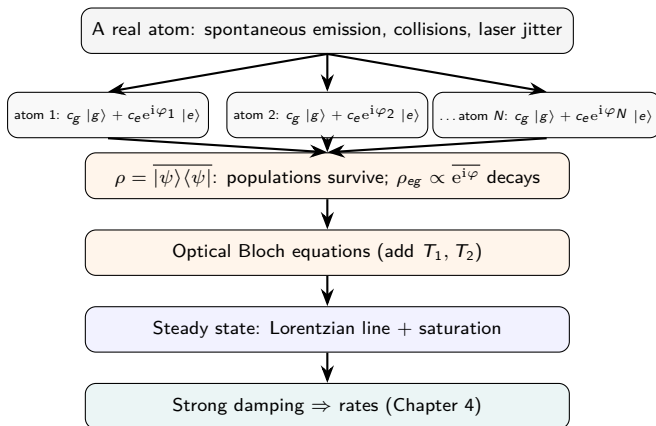
Step 3: dephasing in time. Fully scrambling kicks at rate $1/\tau_c$ (Poisson): the unkicked fraction is e^{-t/τ_c} , so $\rho_{eg}(t) = \rho_{eg}(0)e^{-t/T_2^*}$, $T_2^* = \tau_c$: the $-\rho_{eg}/T_2$ term, with a microscopic face.

Step 4: the bookkeeping object. All needed averages are entries of $\rho = \overline{|\psi\rangle\langle\psi|}$: diagonal = populations, off-diagonal = coherence.

T_1 and T_2 in words

T_1 : energy lifetime ($= 1/\Gamma$ if radiative). T_2 : phase lifetime, $1/T_2 = 1/2T_1 + 1/T_2^*$. **A coherence living T_2 gives a line of width $1/T_2$** (Fourier).

3.1 The ensemble picture, in one diagram



3.1 Where the optical Bloch equations come from

Coherent part + damping, term by term

Step 1: Schrödinger, rewritten. From the rotating-frame amplitudes,

$$\dot{w} = 2 \operatorname{Re}(\dot{c}_e c_e^*) - 2 \operatorname{Re}(\dot{c}_g c_g^*) = -2\Omega \operatorname{Im}(c_e c_g^*) = -\Omega v,$$

and likewise $\dot{u} = \delta v$, $\dot{v} = -\delta u + \Omega w$ (signs: phase conventions; observables indifferent). Compactly $\dot{\mathbf{S}} = \dot{\mathbf{S}}_{\text{eff}} \times \mathbf{S}$, $\dot{\mathbf{S}}_{\text{eff}} = (-\Omega, 0, -\delta)$: a pure **rotation**, $|\mathbf{S}|$ conserved.

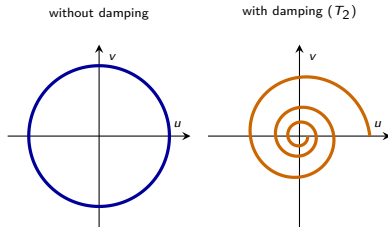
Step 2: damping, one term per process. (i) alone, populations relax to $w_{\text{eq}} (= -1 \text{ since } \hbar\omega_0 \gg k_B T)$: add $-(w - w_{\text{eq}})/T_1$. (ii) the coherence decays at $1/T_2$: add $-u/T_2$, $-v/T_2$. Lindblad (Appendix A) confirms, and gives $T_1 = 1/\Gamma$, $T_2 = 2/\Gamma$ for radiative damping.

Optical Bloch equations

$$\dot{u} = \delta v - \frac{u}{T_2}, \quad \dot{v} = -\delta u + \Omega w - \frac{v}{T_2}, \quad \dot{w} = -\Omega v - \frac{w - w_{\text{eq}}}{T_1}$$

Interpolate from coherent Rabi ($T_{1,2} \rightarrow \infty$) to the rate-equation limit.

3.1 What damping does, seen in the equatorial plane



Precession at δ becomes an inward spiral at $1/T_2$; on the full sphere the vector also drifts to the south pole at $1/T_1$.

T_1 and T_2 are what an MRI scanner measures

Bloch 1946, Nobel 1952. MRI: $\pi/2$ and π RF pulses on water protons; the machine maps T_1 , T_2 voxel by voxel. Fat, grey matter, tumours, CSF relax differently: " T_1 -weighted" vs " T_2 -weighted" images. The Bloch vector spins in every hospital.

3.2 Steady state: solving it, line by line

Solving for the steady state

Driving pumps up, relaxation pulls down; after a few T_1, T_2 they balance: $\dot{u} = \dot{v} = \dot{w} = 0$, $w_{\text{eq}} = -1$.

Step 1: $u = \delta T_2 v$ (in-phase slaved to quadrature). **Step 2:** $v = \frac{\Omega T_2}{1 + \delta^2 T_2^2} w$ (Lorentzian response window $\sim 1/T_2$).

Step 3: close on w : $w = -\frac{1 + \delta^2 T_2^2}{1 + \delta^2 T_2^2 + \Omega^2 T_1 T_2}$. **Step 4:** $\rho_{ee} = (1 + w)/2$: boxed below.

Step 5: power broadening free of charge. As a function of δ : Lorentzian whose denominator doubles at $\delta^2 T_2^2 = 1 + s_0$:

$$\Delta\omega = \frac{2}{T_2} \sqrt{1 + s_0}, \quad s_0 = \Omega^2 T_1 T_2$$

strong drive widens the line: precision spectroscopy works at $s_0 \ll 1$.

Steady-state excited population

$$\rho_{ee}^{\text{ss}} = \frac{1}{2} \frac{s}{1 + s}, \quad s = \frac{\Omega^2 T_1 T_2}{1 + \delta^2 T_2^2}$$

$\Omega \rightarrow \infty$: $\rho_{ee} \rightarrow \frac{1}{2}$. The saturation Lorentz lacked, finally proved.

3.2 Saturation intensity, and the promise kept

Where I_{sat} comes from

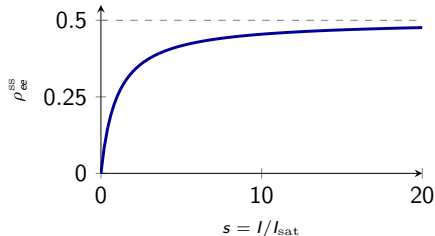
Closed radiative transition: $T_1 = 1/\Gamma$, $T_2 = 2/\Gamma$, so $s_0 = 2\Omega^2/\Gamma^2$. With $\Omega^2 = d^2 E_0^2/\hbar^2$, $I = \frac{1}{2}\epsilon_0 c E_0^2$, and eliminating d via $d^2 = 3\pi\epsilon_0 \hbar c^3 \Gamma/\omega_0^3$:

$$I_{\text{sat}} = \frac{\hbar\omega_0^3\Gamma}{12\pi c^2} = \frac{\pi\hbar c}{3\lambda_0^3\tau}$$

Physically: $I_{\text{sat}}\sigma_0 \sim \hbar\omega_0\Gamma$: one photon per lifetime through the resonant cross section; more light cannot be processed.

Na D₂

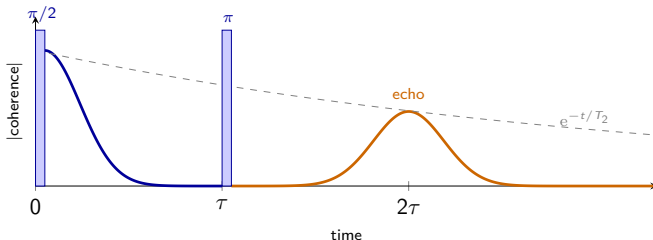
$I_{\text{sat}} \approx 6.2 \text{ mW/cm}^2$. Chapter-1 laser (10 mW/cm^2): $s \approx 1.6$, or via $s = 2\Omega^2/\Gamma^2 = 2(8.7/9.8)^2 \approx 1.6$: the same. Close to saturation, $t_\pi \approx 60 \text{ ns} \sim \tau$: oscillations marginally resolved. The Chapter-1 comparison, now meaningful.



Saturation is the engine of laser cooling

$\sim 10^7$ photon kicks/s: deceleration $\sim 10^5 g$, atoms braked from hundreds of m/s in ms. The $\Gamma/2$ ceiling caps the force and the brightness. Nobel 1997; gateway to BEC and atom arrays.

3.3 Reversible vs irreversible dephasing: the photon echo



Pulses as rotations

Inhomogeneous ensemble: dipoles fan out (T_2^*). A π -pulse at τ flips the fan; fast dipoles now lag: rephased burst at 2τ . Echo height decays with the *true* T_2 : reversible and irreversible dephasing separated. Optical analogue of the Hahn spin echo.

Fighting decoherence; quantum memories

Echo $\rightarrow$ dynamical decoupling (π -pulse trains), standard on real qubits; applied to atomic ensembles it stores light and re-emits on command: a quantum memory for quantum networks. Reversible vs irreversible = recoverable vs lost information.

Check yourself (Chapter 3)

Can you answer these out loud?

- ▶ Why is a single wavefunction no longer enough to describe a real atom?
- ▶ What is the physical difference between T_1 and T_2 ?
- ▶ Why can a strongly driven two-level atom never be more than half excited?
- ▶ Why does a coherence that lives for a time T_2 produce a line of width $1/T_2$?

4. Why do coherent oscillations turn into transition rates?

Oscillation vs rate

In L2 the driven atom *oscillated*; nothing decayed. Yet a spectroscopist measures exactly *rates*: absorption coefficients, photons per second. A rate appears when the atom couples not to one final state but to a **continuum** (many photon directions, many momenta). The result, Fermi's golden rule, is the most reused formula of the course.

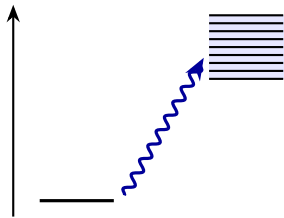
Roadmap, and the symbols introduced here

✓ dissipation $\rightarrow$ (**you are here**) transition rates. New: $W_{i \rightarrow f}$, $\rho(E)$ (density of states), A, B , σ_0 , f (oscillator strength).

Three assumptions, and what “rate” means

(1) weak coupling: first order in V , initial state barely emptied; (2) intermediate times: $1/\omega_0 \ll t \ll$ depletion time (fs to ns: enormous window); (3) a *continuum* of final states, of density ρ ; a single final state gives oscillation, not decay.

4.1–4.2 The setting, and the first-order amplitude



One occupied state, weakly coupled to a dense band of final states.

$H = H_0 + V(t)$; expand $|\psi\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle$:

$$i\hbar \dot{c}_m = \sum_n V_{mn}(t) e^{i\omega_{mn}t} c_n$$

Prepared in $|i\rangle$, weak V , first order ($c_i \approx 1$):

$$c_f^{(1)}(t) = \frac{1}{i\hbar} \int_0^t V_{fi}(t') e^{i\omega_{fi}t'} dt'$$

Two separate questions

Why does a *rate* appear? (Steps 1–4: probability $\propto t$.)
Why the *density of states*? (Step 5: sum over the continuum.)

4.2 First idea: why a rate appears (Steps 1–4)

From the amplitude to linear-in-time probability

Step 1. Harmonic $V(t) = V e^{-i\omega t} + V^\dagger e^{i\omega t}$ (absorption / stimulated emission); keep absorption.

Step 2. Name the mismatch $\Delta = \omega_{fi} - \omega$:

$$c_f^{(1)} = \frac{V_{fi}}{i\hbar} \int_0^t e^{i\Delta t'} dt' = -\frac{V_{fi}}{\hbar} \frac{e^{i\Delta t} - 1}{\Delta}$$

on resonance it grows; off resonance it wobbles.

Step 3. $|e^{i\Delta t} - 1|^2 = 2 - 2 \cos \Delta t = 4 \sin^2(\Delta t/2)$:

$$P_{i \rightarrow f}(t) = \frac{|V_{fi}|^2}{\hbar^2} \frac{\sin^2(\Delta t/2)}{(\Delta/2)^2} \equiv \frac{|V_{fi}|^2}{\hbar^2} f_t(\Delta)$$

Step 4. Height $f_t(0) = t^2$; zeros at $\Delta = \pm 2\pi/t$; area $\int f_t d\Delta = 2\pi t$. Diverging height, vanishing width, finite area:

$$f_t(\Delta) \xrightarrow{t \rightarrow \infty} 2\pi t \delta(\Delta)$$

energy conservation becoming exact; probability $\propto t$: a **rate**.

4.2 Second idea: the density of states; and the golden rule

Step 5: sum over final states

$P_{i \rightarrow f} = \frac{2\pi t}{\hbar^2} |V_{fi}|^2 \delta(\Delta)$; summing over the continuum with $\delta(\Delta) = \hbar \delta(E_f - E_i - \hbar\omega)$:

$$W = \frac{2\pi}{\hbar} |V_{fi}|^2 \rho(E_f) \Big|_{E_f = E_i + \hbar\omega}$$

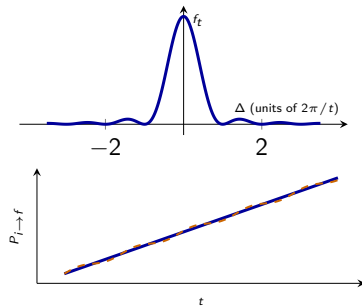
Fermi's golden rule

$$W_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | V | i \rangle|^2 \rho(E_f)$$

The ρ lever is what Purcell will pull (L7).

Validity window (Na D₂)

$10^{-15} \text{ s} \ll t \ll 10^{-8} \text{ s}$: the rate picture is exact for everyday use.



Top: sinc^2 , height t^2 , width $1/t$. Bottom: summed over the continuum, linear growth; its slope is W (dashed: one state alone only oscillates).

4.3 Selection rules: which transitions are allowed

The photon carries angular momentum

Spin-1, odd parity: the electron must absorb both. Hence all the rules below; violated $\Rightarrow \mathbf{d}_{fi} = 0$: “forbidden”.

Electric-dipole selection rules

opposite parity; $\Delta\ell = \pm 1$; $\Delta m = 0, \pm 1$ (0: π -polarised, ± 1 : circular); $\Delta s = 0$; $\Delta j = 0, \pm 1$ (not $0 \rightarrow 0$)

Why spectra are fingerprints

Na $3s \rightarrow 3p$: allowed, bright. $3s \rightarrow 3d$ or $3s \rightarrow 4s$: forbidden, absent. Each element: its own ladder, its own lines.

How we know what stars are made of

Fraunhofer lines at each element's allowed wavelengths; helium found in the Sun before Earth; exoplanet atmospheres read in transit.

The aurora: forbidden lines across the sky

Auroral green (557.7 nm, O): $\tau \approx 0.7$ s, $10^7 \times$ an allowed line; shines only where collisions are seconds apart (~ 100 km up). Red curtain: $\tau \approx 110$ s. “Nebulium” was forbidden oxygen. Selection rules decide what may glow.

4.4 Planck's law, derived (we are about to use it)

Four lines that started quantum mechanics

Step 1: count modes (Appendix A.3): $\rho(\omega) = \omega^2/\pi^2 c^3$ per unit volume; $u(\omega) = \rho(\omega) \times \langle E \rangle_{\text{mode}}$.

Step 2: classical catastrophe. Equipartition $\langle E \rangle = k_B T$: $u \propto \omega^2 k_B T$ diverges: infinite energy in the oven. The counting is impeccable; $\langle E \rangle$ must be wrong.

Step 3: Planck. $E_n = n\hbar\omega$, Boltzmann weights, geometric series $Z = 1/(1 - e^{-x})$, $x = \hbar\omega/k_B T$:

$$\langle n \rangle = -\frac{d \ln Z}{dx} = \frac{1}{e^x - 1}$$

$\hbar\omega \ll k_B T$: back to $k_B T$; $\hbar\omega \gg k_B T$: frozen out. The quantum tames the ultraviolet.

Step 4: assemble.

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

4.4 The B coefficient from first-order perturbation theory

Broadband light: the spectrum is the continuum

Step 1. Thermal light: dense frequencies, random phases; the $\text{sinc}^2 \rightarrow \delta$ mechanism operates, integrated over the spectrum.

Step 2: one slice. $|V_{fi}| = \frac{1}{2} |\mathbf{d} \cdot \boldsymbol{\epsilon}| E_\omega$; isotropic average $\langle |\mathbf{d} \cdot \boldsymbol{\epsilon}|^2 \rangle = d^2/3$; energy density $u(\omega) d\omega = \frac{1}{2} \epsilon_0 E_\omega^2$.

Step 3: add slices incoherently.

$$P(t) = \int \frac{d^2}{3} \frac{2u(\omega)}{4\hbar^2 \epsilon_0} f_t(\omega - \omega_0) d\omega = \frac{\pi d^2}{3\epsilon_0 \hbar^2} u(\omega_0) t$$

$$W_{\text{abs}} = B u(\omega_0), \quad B = \frac{\pi d^2}{3\epsilon_0 \hbar^2}$$

same coefficient for stimulated emission: $B_{12} = B_{21}$ automatic.

Step 4: check to savour. $A = B \cdot \hbar \omega_0^3 / \pi^2 c^3 = \omega_0^3 d^2 / 3\pi \epsilon_0 \hbar c^3$: exactly the Γ of L5. Einstein's 1917 thermodynamics knew the answer before the theory existed.

4.4 Detailed balance fixes A/B

Einstein 1917

Rate equation in thermal light: $\dot{N}_e = -AN_e - BuN_e + BuN_g$.

Step 1. Equilibrium: $\dot{N}_e = 0$, $N_e/N_g = e^{-\hbar\omega_0/k_B T}$.

Step 2–3. Solve: $u(\omega_0) = \frac{A/B}{e^{\hbar\omega_0/k_B T} - 1}$.

Step 4. Match Planck: the Bose factor cancels, valid at *all* T :

$$\boxed{\frac{A}{B} = \frac{\hbar\omega_0^3}{\pi^2 c^3}}$$

Step 5: the lesson. Without A , no equilibrium; semiclassically $A = 0$. Consistency *requires* the quantised vacuum: the bridge to L5.

4.4 The resonant cross section (i): Larmor and the definition

$\sigma_0 = 3\lambda^2/2\pi$, Steps 1–2

A cross section is a target area: power removed $= \sigma \times$ intensity.

Step 1: scattered power. Larmor for $qa \rightarrow |\ddot{\mathbf{p}}| = \omega^2 \mathbf{p}_0$ (structure forced: $\propto q^2 a^2$, dimensions fix $q^2 a^2 / \epsilon_0 c^3$; time average brings $\frac{1}{2}$):

$$P_{\text{sc}} = \frac{\omega^4 |\mathbf{p}_0|^2}{12\pi\epsilon_0 c^3}$$

Step 2: definition. $I = \frac{1}{2}\epsilon_0 c E_0^2$, $\mathbf{p}_0 = \alpha \mathbf{E}_0$:

$$\sigma(\omega) = \frac{P_{\text{sc}}}{I} = \frac{\omega^4 |\alpha(\omega)|^2}{6\pi\epsilon_0^2 c^4}$$

4.4 The resonant cross section (ii): assembling

$$\sigma_0 = 3\lambda^2/2\pi, \text{ Steps 3-6}$$

Step 3: on resonance. $\alpha(\omega_0) = \frac{e^2/m_e}{-i\gamma_c\omega_0}$; $|\alpha|^2 = (e^2/m_e)^2/\gamma_c^2\omega_0^2$.

Step 4: radiative damping, self-consistently. Energy decays exactly as fast as it radiates: $U = \frac{1}{2}m_e\omega_0^2x_0^2$, $P_{\text{rad}} = e^2\omega_0^4x_0^2/12\pi\epsilon_0c^3$:

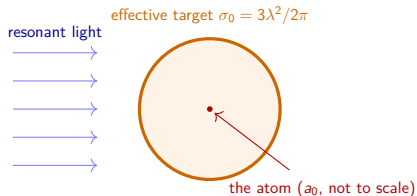
$$\gamma_c = \frac{P_{\text{rad}}}{U} = \frac{e^2\omega_0^2}{6\pi\epsilon_0m_ec^3} \quad (\sim 10^8 \text{ s}^{-1} \text{ optical: off from the true } \Gamma \text{ only by } f \sim 1)$$

Step 5: assemble. Every e, m_e cancels:

$$\sigma_0 = \frac{6\pi c^2}{\omega_0^2} = \frac{6\pi}{k^2} = \frac{3\lambda_0^2}{2\pi}$$

Step 6: degeneracies multiply by $(2J_e + 1)/3(2J_g + 1)$; the scale $\sim \lambda_0^2$ is universal: a weaker dipole gives a *narrower* line of the same peak.

4.4 How big is an atom, as seen by resonant light?



Numbers for Na

$\sigma_0 = 3(589 \text{ nm})^2/2\pi \approx 1.7 \times 10^{-13} \text{ m}^2$: $10^5 \times$ the geometric cross section. Vapour $N = 1 \times 10^{16} \text{ m}^{-3}$: $\alpha = N\sigma_0 \approx 1.7 \times 10^3 \text{ m}^{-1}$: e-folding 0.6 mm.

On resonance the atom fills a region set by the *wavelength*, not by its electron cloud; off resonance the target shrinks by $1/(1 + 4\delta^2/\Gamma^2)$.

4.5 Oscillator strength and the sum rule

$$f_{ge} = \frac{2m_e\omega_0}{\hbar e^2} |\mathbf{d}_{ge}|^2$$

“how many classical electrons a transition is worth”.

Thomas–Reiche–Kuhn sum rule

$$\sum_e f_{ge} = Z$$

A strong line ($f \simeq 1$) exhausts the strength: why one D line dominates sodium.

Γ from f

Substituting $d^2(f)$ into Γ :

$$\Gamma = \frac{e^2\omega_0^2}{6\pi\epsilon_0 m_e c^3} f_{ge} = \gamma_c f_{ge}$$

quantum rate = classical rate $\times f$. At 500 nm, $f \simeq 1$:
 $\Gamma \sim 10^8 \text{ s}^{-1}$, $\tau \sim \text{ns}$.

Proof in four lines

1. $[H, x] = -i\hbar p/m_e \Rightarrow [x, [H, x]] = \hbar^2/m_e$: a pure number.
2. Expand over eigenstates: $\langle g | [x, [H, x]] | g \rangle = 2 \sum_e (E_e - E_g) |x_{eg}|^2$.
3. Equate: $\sum_e f_{ge} = 1$ per electron; Z electrons add.
4. Downward transitions: $\omega_{eg} < 0$, $f < 0$: negative absorption = **gain**, the seed of the laser.

Check yourself (Chapter 4)

Can you answer these out loud?

- ▶ Why does a constant transition *rate* appear only when the final states form a continuum?
- ▶ Where does the density of states $\rho(E)$ come from in the golden rule?
- ▶ Why does a purely classical field predict $A = 0$, and what does that force us to do next?
- ▶ Give one dipole-allowed and one dipole-forbidden transition, and say why.

5. Why can't a classical field explain spontaneous emission?

The atom in the dark

Switch off every lamp and laser: classically $\mathbf{E} = 0$, force zero, nothing should happen. Yet the atom decays in nanoseconds. Something remains in “empty” space: the irreducible fluctuation of the *quantised* field. This is the one place where the semiclassical picture truly breaks.

Roadmap, and the symbols introduced here

✓ rates (Fermi) → **(you are here)** quantised field → spontaneous emission. New: $a, a^\dagger$ (Appendix E), $\mathcal{E}_k = \sqrt{\hbar\omega_k/2\epsilon_0 V}$, Γ , $\tau = 1/\Gamma$, LDOS.

Field in a box: modes $(\mathbf{k}, \lambda)$, each a harmonic oscillator;

$$\hat{\mathbf{E}}(\mathbf{r}) = \sum_{\mathbf{k}\lambda} \mathcal{E}_k \epsilon_{\mathbf{k}\lambda} (a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\lambda}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}})$$

5.1 Where $\mathcal{E}_k$ comes from: energy bookkeeping

Fixing the normalisation

The field energy $H = \int_V (\frac{\epsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}) d^3r$ (electric and magnetic halves equal). Insert one mode: $aa^\dagger$ and $a^\dagger a$ survive the volume integral:

$$H_{\text{field}} = 2\epsilon_0 \mathcal{E}_k^2 V \left(a^\dagger a + \frac{1}{2} \right) \stackrel{!}{=} \hbar\omega_k \left(a^\dagger a + \frac{1}{2} \right) \Rightarrow \mathcal{E}_k = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$$

The zero-point energy $\frac{1}{2}\hbar\omega_k$, spread over V , is the vacuum field squared.

Vacuum field per mode

$$\mathcal{E}_k = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$$

$\langle \mathbf{E} \rangle = 0$, $\langle \mathbf{E}^2 \rangle \neq 0$: this drives spontaneous emission. $\mathcal{E}_k \propto V^{-1/2}$: the key to cavity QED.

The vacuum is not weak

$V = 1 \mu\text{m}^3$, optical ω : $\mathcal{E}_k \approx 1\text{e}5 \text{ V/m}$, within an order of magnitude of a focused mW laser, with zero photons.

5.2 Spontaneous emission = golden rule on the vacuum (i)

No Weisskopf–Wigner machinery needed: an excited atom is one state weakly coupled to the continuum of empty photon modes; the only quantum-field input is $\mathcal{E}_k$.

Steps 1–3

Step 1: matrix element. $V = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$ couples $|e, 0\rangle \rightarrow |g, 1_{\mathbf{k}\lambda}\rangle$; only $a^\dagger$ acts, $\langle 1| a^\dagger |0\rangle = 1$:

$$|\langle g, 1_{\mathbf{k}\lambda} | V | e, 0 \rangle|^2 = \mathcal{E}_k^2 |\mathbf{d} \cdot \epsilon_{\mathbf{k}\lambda}|^2$$

Step 2: sum over all modes. $\Gamma = \frac{2\pi}{\hbar} \sum_{\mathbf{k}\lambda} \mathcal{E}_k^2 |\mathbf{d} \cdot \epsilon|^2 \delta(\hbar\omega_k - \hbar\omega_0)$; box counting: $\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int k^2 dk d\Omega$.

Step 3: directions. Dipole along $\hat{z}$: two polarizations give $d^2 \sin^2 \theta$ (no emission along the axis, the doughnut); $\int \sin^2 \theta d\Omega = 8\pi/3$.

5.2 Spontaneous emission = golden rule on the vacuum (ii)

Steps 4–5

Step 4: magnitude. $\omega_k = ck$, $k^2 dk = \omega_k^2 d\omega_k / c^3$; the δ -function sets $\omega_k = \omega_0$; the box volume V **cancels** (free space cannot depend on an imaginary box).

Step 5: collect. $\frac{2\pi}{(2\pi)^3} \cdot \frac{8\pi}{3} \cdot \frac{1}{2} = \frac{1}{3\pi}$:

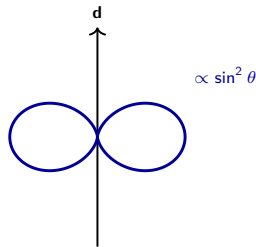
$$\Gamma = \frac{\omega_0^3 d^2}{3\pi\epsilon_0 \hbar c^3}$$

V reappears the moment the mode density is modified: the Purcell effect (L7).

Spontaneous emission rate

$$\Gamma = \frac{\omega_0^3 d^2}{3\pi\epsilon_0 \hbar c^3} = A_{21}$$

$\rho_{ee} = e^{-\Gamma t}$; natural Lorentzian of width Γ ; $\tau = 1/\Gamma$.



The radiation pattern from the polarization sum: zero along the dipole axis, maximal in the equatorial plane; its solid-angle integral gave the $8\pi/3$.

5.2 Decay, linewidth, lifetime

From a constant rate to $e^{-\Gamma t}$ and the Lorentzian

(a) Memoryless continuum: $\dot{\rho}_{ee} = -\Gamma \rho_{ee} \Rightarrow \rho_{ee} = e^{-\Gamma t}$ (Weisskopf–Wigner confirms).

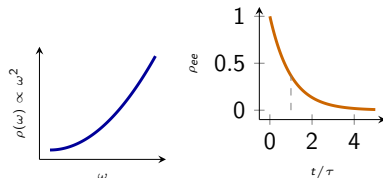
(b) Emitted field $e^{-i\omega_0 t} e^{-\Gamma t/2}$; Fourier:

$$|E(\omega)|^2 \propto \frac{1}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

FWHM Γ : lifetime *is* linewidth; same Fourier pair as the T_2 line of L3.

Lifetime of Na 3p

$$\Gamma = \frac{(3.2 \times 10^{15})^3 (2.1 \times 10^{-29})^2}{3\pi\epsilon_0 \hbar c^3} \approx 6.0 \times 10^7 \text{ s}^{-1}; \quad \tau \approx 17 \text{ ns}, \quad \Gamma/2\pi \approx 9.5 \text{ MHz} \text{ (measured 16.2 ns; scaling } \Gamma \propto \omega_0^3 d^2 \text{ exact).}$$



Why $\Gamma \propto \omega_0^3$

One ω from the photon energy, two from $\rho \propto \omega^2$.
X-rays: fs; optical: ns; 21-cm line: $\tau \sim 10^7$ years.

The Lamb shift: the other face

The same vacuum coupling shifts levels: H $2s_{1/2}$ – $2p_{1/2}$ split ≈ 1 GHz. Decay and shift = Im and Re of one self-energy.

5. Why it matters; check yourself

Γ runs the lit world, and limits qubits

LEDs, OLED pixels, fireflies, phosphors: all spontaneous emission; engineering Γ is a billion-dollar business. For an optical qubit, the same Γ is the fundamental T_1 : hence clock states on *nearly forbidden* transitions (tiny d , tiny Γ , hard to drive: the designer's trade-off).

Can you answer these out loud? (Chapter 5)

- ▶ Why can a classical field never produce spontaneous emission?
- ▶ What is $\mathcal{E}_{\text{vac}}$, and where did we admit it?
- ▶ Why does the spontaneous-emission rate scale as $\Gamma \propto \omega_0^3$?
- ▶ What sets the natural linewidth of a spectral line?

6. How does one atom become a material?

Field and matter must be solved together

A real material is N emitters that all radiate: each feels the incoming wave *plus* what the others re-emitted. The field drives the atoms; the atoms reshape the field. Solved together: a medium with an index, an absorption coefficient, a colour.

Roadmap, and the symbols introduced here

✓ single-atom response $\rightarrow$ **(you are here)** many atoms = a medium. New: $\mathbf{P}$, $\chi(\omega)$, n , α , n_g .

Polarization $\mathbf{P} = N \text{Tr}(\rho \hat{\mathbf{d}}) = Nd(u \cos \omega t - v \sin \omega t)$ feeds the wave equation $\partial_z^2 E - \frac{1}{c^2} \partial_t^2 E = \mu_0 \partial_t^2 P$.

6.1 Closing the loop: the SVEA reduction

The SVEA reduction

Step 1: factor the carrier. $E = \frac{1}{2}\mathcal{E}(z, t)e^{i(kz - \omega t)} + \text{c.c.}$, same for P ; envelopes slow: $|\partial_z \mathcal{E}| \ll k|\mathcal{E}|$.

Step 2: substitute, drop second derivatives of envelopes.

Step 3: watch the cancellation. $-k^2\mathcal{E}$ and $+\omega^2\mathcal{E}/c^2$ cancel exactly ($k = \omega/c$): what survives is *first order*.

$$\left(\partial_z + \frac{1}{c}\partial_t\right)\mathcal{E} = \frac{i\mu_0\omega c}{2}\mathcal{P}$$

Step 4: atomic source. $\mathcal{P} = Nd(u + iv)$; multiply by $d/\hbar$:

Maxwell–Bloch (SVEA, plane wave)

$$\left(\partial_z + \frac{1}{c}\partial_t\right)\Omega = i\frac{\mu_0 c \omega Nd^2}{2\hbar}(u + iv) \quad + \quad \text{Bloch equations for } (u, v, w)$$

The medium shapes the pulse; the pulse drives the medium.

“Two-level emitter” usually means a piece of solid

NV centres, rare-earth ions, quantum dots: a d and a T_2 in a crystal. Read “ N atoms” as “ N defects or dots”.

6.2 The susceptibility from the steady-state coherence

Weak probe on a dilute medium

Step 1: weak field. $w \simeq -1$ freezes the inversion; the coherence obeys

$$\dot{\tilde{\rho}}_{eg} = -\left(\frac{1}{T_2} - i\delta\right)\tilde{\rho}_{eg} + \frac{i\Omega}{2}$$

Step 2: steady state. $\tilde{\rho}_{eg}^{\text{ss}} = \frac{i\Omega/2}{1/T_2 - i\delta} = -\frac{\Omega/2}{\delta + i/T_2}$.

Step 3: one atom $\rightarrow$ medium. $P = 2Nd\tilde{\rho}_{eg}^{\text{ss}} \equiv \varepsilon_0\chi E$, $\Omega = dE/\hbar$.

Step 4: read off. $\chi = -\frac{Nd^2}{\varepsilon_0\hbar} \frac{\delta - i/T_2}{\delta^2 + 1/T_2^2}$: with $e^{-i\omega t}$, $\chi'' > 0$ comes out automatically, as absorption must.

Absorption and dispersion of N emitters

$$\chi' = -\frac{Nd^2}{\varepsilon_0\hbar} \frac{\delta}{\delta^2 + 1/T_2^2}, \quad \chi'' = \frac{Nd^2}{\varepsilon_0\hbar} \frac{1/T_2}{\delta^2 + 1/T_2^2}$$

Lorentz's χ with $\gamma_c \rightarrow 1/T_2$. Beer's law: $I = I_0 e^{-\alpha z}$, $\alpha = \omega\chi''/c$; width exactly $1/T_2$.

6.2 Applications: memories, superradiance, absorption lengths

Storing light in a crystal

Rare-earth ions (Eu, Er): T_2 up to ms, kHz-narrow lines in a solid; control pulses freeze and release the collective coherence: a **quantum memory**. Erbium sits at $1.5\,\mu\text{m}$: plugs into telecom fibre.

Why superradiance scales as N^2

Random phases: intensities add, $I \propto N$. Phased dipoles in $V \ll \lambda^3$: *amplitudes* add, $I_{\text{max}} \propto N^2$; energy $N\hbar\omega_0 \Rightarrow$ burst time $\sim 1/N\Gamma$ (Dicke: builds from noise after $\ln N/N\Gamma$).

Dicke superradiance

$$\Gamma_N = N\Gamma, \quad I_{\text{max}} \propto N^2 \hbar\omega_0 \Gamma$$

Two numbers

Burst: $N = 10^4$, $\Gamma = 6\text{e}7\text{s}^{-1}$: $\Gamma_N \approx 6\text{e}11\text{s}^{-1}$ (1.7 ps). Rb vapour on resonance ($N = 1\text{e}17\text{m}^{-3}$): $\chi'' \approx 3.6 \times 10^{-3}$, $\alpha \approx 2.9\text{e}4\text{m}^{-1}$: absorption length $35\,\mu\text{m}$.

Beyond linear

Full Maxwell–Bloch: self-induced transparency (2π soliton), photon echoes, superradiance.

6.3 Group velocity and slow light

Group index in three lines

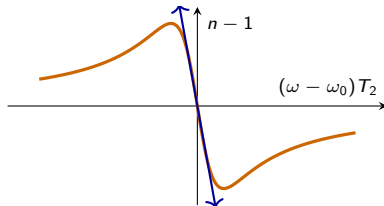
A pulse travels at $v_g = d\omega/dk$; $k = n(\omega)\omega/c$:

$$v_g = \frac{c}{n + \omega \, dn/d\omega} \equiv \frac{c}{n_g}$$

n stays near 1; it is the *slope*, times $\omega \sim 10^{15}$, that makes n_g huge.

How slow is slow?

$dn/d\omega \sim 10^{-7} \text{ s}$ near a sharp line: $n_g \sim 2 \times 10^8$, $v_g \approx 1.5 \text{ m/s}$: walking pace. First done in cold atoms; now engineered in rare-earth crystals and semiconductors.



Optical buffers and the quantum internet

A pulse crawling at m/s is *stored*: photonic routers without electronics; stopping light in a crystal is how quantum memories catch a photonic qubit.

6. The telecom window; check yourself

Why the world's data travels at $1.55\text{ }\mu\text{m}$

Silica's χ'' is pinched between Rayleigh scattering ($\propto \omega^4$, Chapter 1) and the Si–O phonon wing (Chapter 8): minimum $\approx 0.2\text{ dB/km}$: 1% of the power survives 100 km (window glass loses that in metres). Erbium amplifies exactly there: hence the fibre internet.

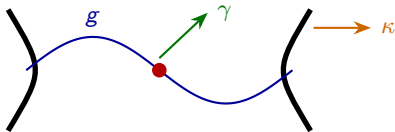
Can you answer these out loud? (Chapter 6)

- ▶ Why must the field and the atoms be solved self-consistently in a medium?
- ▶ What does χ' control, and what does χ'' control?
- ▶ Why does a sharp resonance slow light down?
- ▶ Why is the width of the absorption line equal to $1/T_2$?

7. Can we modify spontaneous emission?

Γ is not a constant of nature

Γ was Fermi's rule $\times$ the density of photon modes. The dipole is fixed; the mode density depends on the *surroundings*. Change them (a mirror box) and Γ changes: the Purcell effect, one of photonics' most useful knobs.



Coupling g ; photon leak κ ; free-space leak γ . Strong coupling: $g \gg \kappa, \gamma$.

Roadmap; what you actually need

New: Q , V , g , κ , γ , F_P , $C = g^2/\kappa\gamma$. The core result is the **Purcell factor** (demonstration 6 of the thread): golden rule with the cavity mode count. Jaynes–Cummings and dressed states are enrichment: intuition and the strong-coupling regime.

7.1 Building the Jaynes–Cummings Hamiltonian

Two pieces of notation

$\sigma = |g\rangle\langle e|$ (de-excite), $\sigma^\dagger = |e\rangle\langle g|$ (excite), $\sigma^\dagger\sigma$ counts excitation; $a, a^\dagger$ for the photon (Appendix E).

Assembled from three known pieces

Step 1: atom. $H_{\text{atom}} = \hbar\omega_0\sigma^\dagger\sigma$. **Step 2: mode.** $H_{\text{field}} = \hbar\omega_c a^\dagger a$.

Step 3: coupling. $-\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$ with $\hat{\mathbf{d}} = d(\sigma + \sigma^\dagger)$, $\hat{\mathbf{E}} \propto (a + a^\dagger)$:

$$(\sigma + \sigma^\dagger)(a + a^\dagger) = \underbrace{\sigma^\dagger a}_{\text{absorb}} + \underbrace{\sigma a^\dagger}_{\text{emit}} + \underbrace{\sigma^\dagger a^\dagger + \sigma a}_{\text{2 quanta at once}}$$

Step 4: RWA. Drop the two-quanta terms (oscillate at 2ω).

Jaynes–Cummings

$$H = \hbar\omega_0\sigma^\dagger\sigma + \hbar\omega_c a^\dagger a + \hbar g(\sigma^\dagger a + \sigma a^\dagger)$$

$$g = d\sqrt{\frac{\omega_c}{2\hbar\epsilon_0 V}}$$

g : dipole $\times$ single-photon field; $g \propto V^{-1/2}$.

7.1 Dressed states and vacuum Rabi splitting

Diagonalising one block

Step 1. $\hat{N} = \sigma^\dagger \sigma + a^\dagger a$ commutes with H : 2×2 blocks $\{|e, n\rangle, |g, n+1\rangle\}$.

Step 2. $N = 1$, resonance: $H = \hbar \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$.

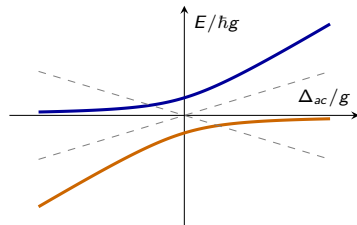
Step 3. $\lambda = \pm \hbar g$: $|\pm\rangle = (|e, 0\rangle \pm |g, 1\rangle)/\sqrt{2}$, $E_\pm = \hbar\omega_c \pm \hbar g$.

Step 4. From $|e, 0\rangle$: $P_e(t) = \cos^2(gt)$: *vacuum Rabi oscillations*: the quantum swaps with the *empty* cavity.

Vacuum Rabi splitting

$$E_\pm = \hbar\omega_c \pm \hbar g, \quad \text{splitting } 2\hbar g$$

The doublet is the signature of strong coupling.



Avoided crossing of the dressed states (solid) vs bare levels (dashed); gap $2\hbar g$ at resonance.

Circuit QED

Transmon + microwave resonator = this Hamiltonian. Strong coupling swaps quantum information; the dispersive pull reads the qubit out. The doublet is the beating heart of a quantum processor.

7.1 The Jaynes–Cummings ladder; blockade; collapse and revival



Each n -manifold splits by $2g\sqrt{n+1}$: an **anharmonic** ladder (a classical field would give equal rungs).

- ▶ **Photon blockade:** absorbing one photon detunes the next step: a single-photon turnstile.
- ▶ **Collapse and revival:** a coherent field's spread of n dephases and rephases the Rabi oscillation: direct evidence of field quantisation.

Weak vs strong coupling

$g \gg \kappa, \gamma$: reversible exchange, dressed states, a new quantum object. $g \ll \kappa$: the cavity is a modified continuum: emission stays irreversible but its *rate* changes: the Purcell regime.

7.3 The Purcell factor: the core result

Golden rule twice, then the ratio

Step 1: cavity mode density. One mode, Lorentzian of width $\kappa = \omega_c/Q$, normalised to $\int \rho d\omega = 1$:

$$\rho_{\text{cav}}(\omega) = \frac{1}{\pi} \frac{\kappa/2}{(\omega - \omega_c)^2 + (\kappa/2)^2} \Rightarrow \rho_{\text{cav}}(\omega_c) = \frac{2}{\pi \kappa} = \frac{2Q}{\pi \omega_c}$$

higher Q : narrower mode, larger on-resonance density.

Step 2: cavity rate (emitter at antinode, aligned): $\Gamma_{\text{cav}} = \frac{2\pi}{\hbar} d^2 \frac{\hbar \omega_c}{2\varepsilon_0 V} \frac{2Q}{\pi \omega_c} = \frac{2d^2 Q}{\hbar \varepsilon_0 V}$.

Step 3: free space. $\Gamma_{\text{free}} = \omega_c^3 d^2 / 3\pi \varepsilon_0 \hbar (c/n)^3$.

Steps 4–5: ratio; $d, \hbar, \varepsilon_0$ **cancel;** $(c/n)/\omega_c = \lambda_c/2\pi n$:

Purcell factor

$$F_P = \frac{\Gamma_{\text{cav}}}{\Gamma_{\text{free}}} = \frac{3}{4\pi^2} \left(\frac{\lambda_c}{n} \right)^3 \frac{Q}{V}$$

Enhanced on resonance; *inhibited* off resonance or in a photonic band gap. The LDOS is now an engineering parameter.

7.3–7.4 Purcell in numbers; adiabatic elimination; cooperativity

Micropillar single-photon source

$Q = 10^4$, $V = 10(\lambda/n)^3$, $\lambda_c = 950$ nm, $n = 3.5$: $F_P \approx 76$. Dot with $\tau_0 = 1$ ns: emits in 13 ps; fraction $\beta = F_P/(F_P+1) \approx 0.99$ into one mode. (Diffraction-limited $V = (\lambda/n)^3$ would give 760.)

Strong-coupling threshold

$g/2\pi \sim 20$ GHz; $\kappa/2\pi = \nu_c/Q \approx 32$ GHz; $\gamma/2\pi \sim 0.2$ GHz: at the onset; $Q \rightarrow 10^5$ reveals the doublet.

Purcell-engineered light

On-demand single photons for quantum communication; faster LEDs and nanolasers; suppressed emission for efficiency. All one knob: $(\lambda/n)^3 Q/V$.

Adiabatic elimination ($\kappa \gg g$)

One-excitation amplitudes: $\dot{c}_e = -igc_c$, $\dot{c}_c = -igc_e - (\frac{\kappa}{2} + i\Delta_{ac})c_c$. Fast c_c follows: $c_c \simeq -igc_e/(\kappa/2 + i\Delta_{ac})$; substitute:

$$\Gamma_{\text{cav}} = \frac{4g^2}{\kappa} \frac{1}{1 + (2\Delta_{ac}/\kappa)^2}$$

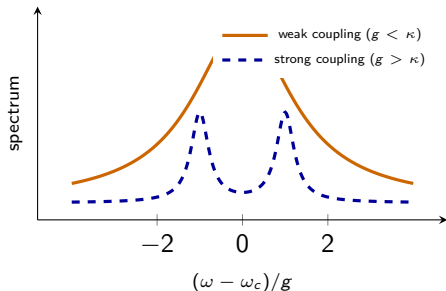
(imaginary part = the dispersive shift used for qubit readout). With $g^2 \propto 1/V$, $\kappa = \omega_c/Q$: exactly F_P : the two viewpoints agree.

Cooperativity

$$C = \frac{g^2}{\kappa\gamma} \quad (\simeq F_P \text{ in weak coupling})$$

$C \gg 1$: efficient single-photon devices, atom–photon gates.

7. Emission spectrum; check yourself



Raising Q : from the Purcell-broadened line to the vacuum Rabi doublet.

Can you answer these out loud? (Chapter 7)

- ▶ Why is Γ *not* a fixed constant of nature?
- ▶ In one sentence, how does a cavity change Γ ?
- ▶ What roles do Q and V play in the Purcell factor?
- ▶ Why do we *not* need the dressed states to obtain the Purcell factor?

8. What changes when one atom becomes 10^{23} ?

From levels to bands

Atoms bonded so close that their levels overlap and broaden into *bands*. The coupling $-\mathbf{d} \cdot \mathbf{E}$ is unchanged, but “initial/final state” means valence/conduction band, and the golden-rule sum becomes an integral over a continuum. The answer: the optical response of every semiconductor device you own.

Roadmap, and the symbols introduced here

✓ one emitter $\rightarrow$ **(you are here)** 10^{23} = a solid. New: E_g , $J(\hbar\omega)$ (joint DOS), $\varepsilon = \varepsilon_1 + i\varepsilon_2$, Ry^* , ω_p , ω_T , ω_L .

The longest lecture: bands $\rightarrow$ dielectric function $\rightarrow$ Kramers–Kronig $\rightarrow$ absorption edge $\rightarrow$ excitons $\rightarrow$ polaritons $\rightarrow$ Drude $\rightarrow$ microcavity polaritons.

8.1 From atomic levels to energy bands

Tight-binding band from a chain

Step 1: Bloch ansatz. $|\psi_k\rangle = \frac{1}{\sqrt{N}} \sum_n e^{ikna} |n\rangle$.

Step 2: Hamiltonian. On-site E_0 , hopping $-t$ to neighbours.

Step 3: shift the dummy index: $|\psi_k\rangle$ factors out.

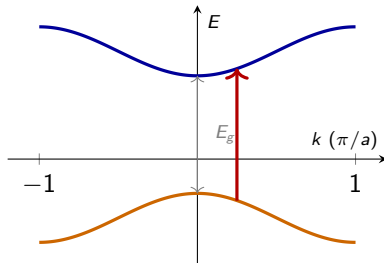
Step 4: dispersion.

$$E(k) = E_0 - 2t \cos(ka)$$

band of width $4t$; near the bottom parabolic, $m^* = \hbar^2/2ta^2$.

Why optical transitions are vertical

Photon momentum $\hbar\omega/c \sim 10^{-27}$ vs zone-edge $\hbar\pi/a \sim 10^{-24}$ kg m/s: $\mathbf{k}_c \approx \mathbf{k}_v$. The dipole approximation in crystal language.



Valence (orange) and conduction (blue) bands; red arrow: a vertical optical transition; E_g sets the absorption threshold.

8.2 The dielectric function, and ε_2 from the golden rule

The recurrence made visible: the same denominator $\omega_0^2 - \omega^2 - i\gamma\omega$ from Chapter 1 (classical), Chapter 6 (quantum $1/(\delta + i/T_2)$), now builds $\varepsilon(\omega)$ of a solid. Free carriers: Drude term; interband transitions: Lorentz oscillators.

Microscopic ε_2

Step 1: absorbed power. Work on bound charges, $\langle \dot{\mathbf{P}} \cdot \mathbf{E} \rangle$; only the quadrature survives the average: $\langle P_{\text{abs}} \rangle = \frac{1}{2} \varepsilon_0 \omega \varepsilon_2 |\mathbf{E}_0|^2$ (χ' borrows and returns; χ'' works continuously).

Step 2: same power from the golden rule: each $v \rightarrow c$ removes $\hbar\omega$ at rate W_{vc} .

Step 3: equate:

$$\varepsilon_2(\omega) = \frac{\pi e^2}{\varepsilon_0 m_e^2 \omega^2} \sum_{v,c} |\mathbf{e} \cdot \mathbf{p}_{cv}|^2 \delta(E_c - E_v - \hbar\omega)$$

matrix element $\times$ joint density of states; ε_1 then follows from causality.

8.2 Kramers–Kronig from causality

Causality alone

Step 1: cause before effect. $\chi(t) = 0$ for $t < 0$ (the medium cannot polarise before the kick).

Step 2: one-sided in time $\Rightarrow$ analytic in frequency. $\chi(\omega) = \int_0^\infty \chi(t)e^{i\omega t}dt$ converges even better for $\text{Im } \omega > 0$: no poles in the upper half plane.

Step 3: Cauchy + contour along the real axis: $\chi(\omega) = \frac{1}{i\pi} \mathcal{P} \int \frac{\chi(\omega')}{\omega' - \omega} d\omega'$: the value at ω is welded to all other frequencies.

Step 4: split re/im and fold onto $[0, \infty)$:

Kramers–Kronig relations

$$\varepsilon_1(\omega) - 1 = \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega' \varepsilon_2(\omega')}{\omega'^2 - \omega^2} d\omega', \quad \varepsilon_2(\omega) = -\frac{2\omega}{\pi} \mathcal{P} \int_0^\infty \frac{\varepsilon_1(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Measure absorption at all frequencies, *compute* the index: ellipsometry's daily bread. Any absorption peak carries anomalous dispersion across it (Exercise 7.1: a δ -line gives exactly the Lorentz shape).

8.3 Interband absorption: the $\sqrt{\cdot}$ edge

Joint density of states

Step 1: cost of a vertical transition. Parabolic bands: $\hbar\omega = E_g + \hbar^2 k^2 / 2\mu$, $1/\mu = 1/m_c + 1/m_v$ (a reduced mass, as in two-body motion).

Step 2: resonant k : $k(\omega) = \sqrt{2\mu(\hbar\omega - E_g)}/\hbar$.

Step 3: count states on the shell ($\times 2$ spin), Jacobian $d(\hbar\omega)/dk = \hbar^2 k/\mu$: $J = \mu k / \pi^2 \hbar^2$.

Step 4: insert $k(\omega)$:

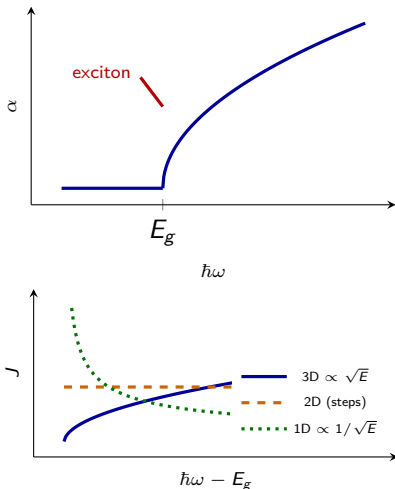
$$J(\hbar\omega) = \frac{1}{2\pi^2} \left(\frac{2\mu}{\hbar^2} \right)^{3/2} \sqrt{\hbar\omega - E_g} \quad \Rightarrow \quad \alpha(\omega) \propto (\hbar\omega - E_g)^{1/2}$$

few resonant electrons at the gap: absorption switches on gently. 2D well: staircase; 1D wire: $1/\sqrt{\cdot}$ spikes.

Direct-gap absorption edge

$\alpha \propto (\hbar\omega - E_g)^{1/2}$ (direct allowed). Indirect (Si, Ge): phonon assistance, $(\hbar\omega - E_g \pm \hbar\omega_{\text{ph}})^2$: weak. Below E_g : transparent.

8.3 The edge in pictures; the gap that lights your room



The band gap that lights your room and powers your roof

E_g picks the colour and the job: blue LED = GaN (2014 Nobel: the missing colour for white light); solar cells run the edge backwards (E_g sets the harvested band); GaAs (direct) emits efficiently: fibre lasers; Si (indirect) barely emits: your computer is Si, its data travels as GaAs light.

8.4 Excitons: the hydrogen atom inside a crystal

Wannier exciton Rydberg series

Step 1: two-body problem. Electron + hole, Coulomb screened by ϵ_r , reduced mass μ .

Step 2: map onto hydrogen ($m_e \rightarrow \mu$, $e^2 \rightarrow e^2/\epsilon_r$): $E_n = -\frac{\mu/m_e}{\epsilon_r^2} \frac{\text{Ry}}{n^2}$.

Step 3: exciton Rydberg and Bohr radius.

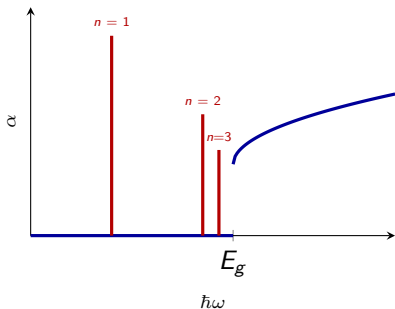
$$E_n = E_g - \frac{\text{Ry}^*}{n^2}, \quad \text{Ry}^* = \frac{\mu/m_e}{\epsilon_r^2} \text{Ry}, \quad a_X = \frac{\epsilon_r m_e}{\mu} a_0$$

Step 4: why giant and weak. $\epsilon_r \sim 10$, $\mu \sim 0.1m_e$: $\text{Ry}^* \sim 10 \text{ meV}$, $a_X \sim 100a_0 \sim 5 \text{ nm}$: many unit cells: the Wannier picture is self-consistent (molecular crystals, $\epsilon_r \sim 1$: Frenkel).

GaAs vs ZnO

GaAs ($\mu = 0.059m_e$, $\epsilon_r = 12.9$): $\text{Ry}^* \approx 4.8 \text{ meV}$, $a_X \approx 12 \text{ nm}$: below $\sim 50 \text{ K}$ only; line at $E_g - E_b = 1.4158 \text{ eV}$ (876 nm). ZnO: $\text{Ry}^* \approx 40\text{--}60 \text{ meV} > k_B T$: room-temperature excitons and lasing.

8.4 The Elliott spectrum; qubits made of crystal



Rydberg series accumulating at E_g ; continuum enhanced by the Coulomb (Sommerfeld) factor; lines resolved when $k_B T \lesssim \text{Ry}^*$.

Qubits made of crystal

Quantum dots: artificial atoms, tunable gap, today's brightest single-photon sources. NV centres in diamond: spins initialised, rotated, read optically at room temperature: qubits and nanoscale magnetometers. Silicon spin qubits piggyback on the chip industry. Band physics says where the levels sit; Lectures 1–7 say how to talk to them.

8.5 Polaritons in bulk: dispersion and the LST relation

One resonance + light

Step 1. TO phonon at ω_T (undamped Lorentz):

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{(\varepsilon_{\text{st}} - \varepsilon_\infty)\omega_T^2}{\omega_T^2 - \omega^2}.$$

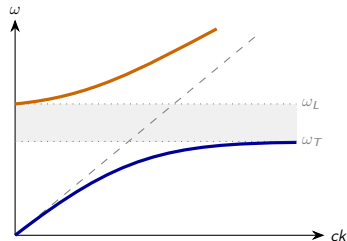
Step 2: transverse modes. $k^2 c^2 = \omega^2 \varepsilon(\omega)$: two branches; lower saturates at ω_T ($\varepsilon \rightarrow \infty$), upper starts where $\varepsilon = 0$.

Step 3: longitudinal mode. $\varepsilon(\omega_L) = 0$: $\omega_L^2 = \omega_T^2 \varepsilon_{\text{st}} / \varepsilon_\infty$.

Step 4:

$$\frac{\omega_L^2}{\omega_T^2} = \frac{\varepsilon_{\text{st}}}{\varepsilon_\infty}$$

between ω_T and ω_L : $\varepsilon < 0$, total reflection: the *reststrahlen* band; its width measures the bond's ionicity.



Lower/upper polariton branches; light line dashed; shaded reststrahlen gap: no propagation. The bulk sibling of the vacuum Rabi splitting.

8.6 Free carriers: Drude, plasma edge, and the colour of things

Drude in three steps

Step 1. Free electron, collisions γ : $\mathbf{r}_0 = \frac{e/m_e}{\omega^2 + i\gamma\omega} \mathbf{E}_0$.

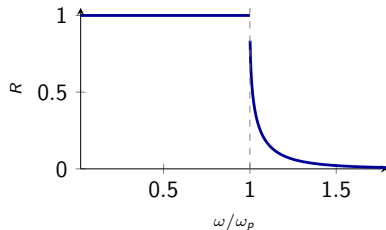
Step 2. $\mathbf{P} = -Ne\mathbf{r}_0$:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}, \quad \boxed{\omega_p^2 = \frac{Ne^2}{\varepsilon_0 m_e}}$$

Step 3. $\omega < \omega_p$: $\varepsilon < 0$, evanescent, $R \rightarrow 1$; $\omega > \omega_p$: transparent. **Step 4 (Hagen–Rubens).** $\omega \ll \gamma$: $\varepsilon \approx i\sigma_0/\varepsilon_0\omega$: Ohmic absorption, skin effect.

Aluminium

$N = 1.8 \times 10^{29} \text{ m}^{-3}$: $\hbar\omega_p \approx 15 \text{ eV}$ (UV): mirror across the whole visible.



Why gold is yellow, silver white, glass transparent

Ag: interband in the UV: white mirror. Au: 5d transitions absorb from $\approx 2.4 \text{ eV}$ (green-blue): yellow reflection; Cu lower still: red. Glass: no free carriers, gap $\sim 9 \text{ eV}$: nothing to absorb, only $n \approx 1.5$. The colour of every object states where its oscillators sit.

8.7 Microcavity exciton-polaritons: closing the loop with Lecture 7

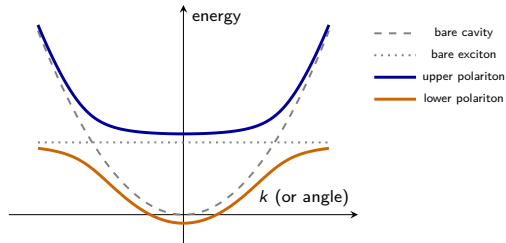
Cavity-polariton anticrossing

Step 1: two coupled modes. Cavity photon $E_c(k)$ (parabolic, tunable by angle) + flat exciton E_X ; coupling $\hbar\Omega_R/2$ (collective, $\propto \sqrt{N}$).

Step 2: eigenvalues.

$$E_{\pm}(k) = \frac{E_c + E_X}{2} \pm \frac{1}{2} \sqrt{(E_c - E_X)^2 + (\hbar\Omega_R)^2}$$

Step 3: anticrossing. At resonance: split by $\hbar\Omega_R$ (normal-mode splitting); eigenstates half-light/half-matter: upper and lower polaritons.



Jaynes–Cummings physics in a chip, with a collective exciton as the atom.

GaAs microcavity

$\hbar\Omega_R \approx 4 \text{ meV}$ (1 well) to $15 \text{ meV} \gg$ linewidths ($\sim 0.5 \text{ meV}$): firmly strong coupling. Polariton mass $\sim 10^{-4} m_e$: condensation at tens of K; GaN, ZnO, organics: room temperature.

Check yourself (Chapter 8)

Can you answer these out loud?

- ▶ In a solid, what plays the role of the “initial” and “final” states of the golden rule?
- ▶ Why does a direct-gap absorption edge rise as $\sqrt{\hbar\omega - E_g}$?
- ▶ Why is silicon a poor light emitter but a fine solar cell?
- ▶ What is an exciton, and why is it “hydrogen inside a crystal”?

Synthesis: one coupling, five tools

Tool	Keeps / adds	Answers the question
Fermi golden rule	rates, density of states	How fast does the atom absorb / emit?
Optical Bloch eqs.	coherence, T_1 , T_2	How does a driven atom evolve in time?
Maxwell-Bloch	field propagation	How does light traverse a resonant medium?
Purcell / cavity QED	engineered mode density	How does the environment reshape emission?
Solid-state response	continuum of states	How does bulk matter respond optically?

The whole course in one sentence

One interaction, $-\mathbf{d} \cdot \mathbf{E}$; one recurring shape, the resonant denominator whose imaginary part is absorption; one recurring lever, the density of states: of final atomic levels (Fermi), of photon modes (spontaneous emission, Purcell), of electronic states (solids).

Problem 1: two-photon vs one-photon rates

Statement. Show that the ratio of two-photon to one-photon transition rates at intensity I scales as I , and estimate where they become comparable.

Solution 1

One-photon: first order, $W^{(1)} \propto |V|^2 \propto I$. Two-photon: second order through a virtual state, $W^{(2)} \propto |V|^4 \propto I^2$, with denominator $(\hbar\Delta)^2$:

$$\frac{W^{(2)}}{W^{(1)}} \sim \frac{I d^2}{\epsilon_0 c \hbar^2 \Delta^2}$$

With $d \sim ea_0$, $\hbar\Delta \sim 1\text{eV}$: equality at $I \sim 10^{11}\text{W/cm}^2$: multiphoton physics needs pulsed lasers: nonlinear optics.

Problem 2: hydrogen $2p \rightarrow 1s$ lifetime

Statement. $d_{2p,1s} = 0.74 ea_0$, $\lambda_0 = 121.6 \text{ nm}$ (Lyman- α): compute τ .

Solution 2

$\omega_0 = 2\pi c/\lambda_0 = 1.55\text{e}16 \text{ rad/s}$, $d = 6.3\text{e} - 30 \text{ C m}$:

$$\Gamma = \frac{\omega_0^3 d^2}{3\pi\epsilon_0 \hbar c^3} \approx 6.3\text{e}8 \text{ s}^{-1} \Rightarrow \tau \approx 1.6 \text{ ns}$$

matching the textbook value: the ω_0^3 factor beats the smaller dipole (vs Na).

Problem 3: steady-state fluorescence

Statement. Find $R_{\text{sc}} = \Gamma \rho_{ee}^{\text{ss}}(\delta, I)$ and its maximum.

Solution 3

$$R_{\text{sc}} = \frac{\Gamma}{2} \frac{I/I_{\text{sat}}}{1 + I/I_{\text{sat}} + (2\delta/\Gamma)^2}$$

Power-broadened Lorentzian; saturates at $\Gamma/2$. Na: $\approx 3\text{e}7$ photons/s per atom; maximum light force $F_{\text{max}} = \hbar k \Gamma/2$: the laser-cooling ceiling.

Problem 4: Purcell engineering for a single-photon source

Statement. What Q/V gives $\beta = 0.95$ for a dot ($\tau_0 = 1$ ns, $\lambda = 900$ nm, $n = 3.5$)?

Solution 4

$\beta = F_P/(F_P + 1) = 0.95 \Rightarrow F_P = 19$. With $V = (\lambda/n)^3$: $Q = 4\pi^2 F_P/3 \approx 250$: very modest. Realistic $V = 5(\lambda/n)^3$: $Q \approx 1250$. Lesson: collection efficiency does not need ultrahigh Q ; small V does the work.

Problem 5: plasma edge of doped InSb

Statement. $N = 1\text{e}24\text{ m}^{-3}$, $m^* = 0.014m_e$, $\epsilon_\infty = 16$: find λ_p .

Solution 5

$$\omega_p = \sqrt{\frac{Ne^2}{\epsilon_0\epsilon_\infty m^*}} \approx 1.2\text{e}14\text{ rad/s} \Rightarrow \lambda_p \approx 16\text{ }\mu\text{m (mid-IR)}$$

Small m^* and low N push the edge from the UV (metals) into the infrared: tunable IR mirrors, plasmonics.

Problem 6: oscillator strength of the Na D line

Statement. Estimate f from $\tau = 16 \text{ ns}$, $\lambda_0 = 589 \text{ nm}$.

Solution 6

Invert $\Gamma = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 m_e c^3} f$:

$$f = \frac{6\pi\epsilon_0 m_e c^3}{e^2 \omega_0^2} \Gamma \approx 0.98$$

The D line exhausts the whole valence electron ($\sum f = 1$): why the two-level model of sodium works so well.

Problem 7: Ramsey fringes

Statement. Two $\pi/2$ -pulses separated by T at detuning δ : find $P_e(\delta)$.

Solution 7

1. First $\pi/2$: south pole $\rightarrow$ equator (unit coherence).
2. Free precession: phase δT about the vertical axis.
3. Second $\pi/2$: projects the precessed vector; the two amplitude paths interfere:

$$P_e(\delta) = \cos^2\left(\frac{\delta T}{2}\right)$$

4. Fringes of period $2\pi/T$: longer T , finer fringes: the atomic clock; contrast damped by e^{-T/T_2} .

Problem 8: optical-trap depth

Statement. Red-detuned Gaussian beam (P , w_0 , δ): trap depth.

Solution 8

1. $I_0 = 2P/\pi w_0^2$. 2. $U_{\text{dip}} = \frac{3\pi c^2 \Gamma}{2\omega_0^3} \frac{1}{\delta} I_0$ (< 0 for red). 3. Na, $P = 10$ mW, $w_0 = 1 \mu\text{m}$, $\delta/2\pi = -1\text{e}13$ Hz: $I_0 \approx 6.4\text{e}9$ W/m²,

$$|U_{\text{dip}}|/k_B \approx 6 \text{ mK}$$

4. Scattering $\propto I/\delta^2$: suppressed by $1/\delta$: deep trap, few photons: the tweezer principle.

Problem 9: reststrahlen width from LST

Statement. GaAs: $\omega_T/2\pi = 8.0 \text{ THz}$, $\epsilon_{\text{st}} = 12.9$, $\epsilon_{\infty} = 10.9$: find ω_L and the band.

Solution 9

$$\omega_L = \omega_T \sqrt{\epsilon_{\text{st}}/\epsilon_{\infty}} = 8.7 \text{ THz}$$

Total reflection for $8.0 \text{ THz} < \omega/2\pi < 8.7 \text{ THz}$ ($\approx 3 \text{ meV}$, $\lambda \approx 35 \mu\text{m}$). A macroscopic reflectivity measurement yields the ionicity of the Ga-As bond.

Problem 10: quantum-well absorption staircase

Statement. Show the 2D joint density of states is constant, hence a staircase spectrum.

Solution 10

1. 2D counting: $\mathcal{N} = k^2/2\pi$ per unit area; $\hbar\omega - E_g = \hbar^2 k^2/2\mu$:

$$J_{2D} = \frac{d\mathcal{N}}{d(\hbar\omega)} = \frac{\mu}{\pi\hbar^2} = \text{const}$$

2. Each subband n turns on at $E_g + E_n^{\text{conf}}$: a staircase. 3. Sharp, temperature-robust edges (+ up to $4\times$ exciton binding): why quantum-well lasers and modulators beat bulk.

Problem 11: strong coupling for a quantum dot

Statement. $d = 1\text{e} - 28\text{ C m}$, $V = (\lambda/n)^3$, $\lambda = 950\text{ nm}$, $n = 3.5$; compare g with $\kappa/2\pi = 10\text{ GHz}$.

Solution 11

1. $\lambda/n = 271\text{ nm}$: $V \approx 2.0\text{e} - 20\text{ m}^3$. 2. $g = d\sqrt{\omega_c/2\hbar\epsilon_0 V}$ with $\omega_c = 2.0\text{e}15\text{ rad/s}$: $g \approx 7.3\text{e}11\text{ rad/s}$, $g/2\pi \approx 120\text{ GHz}$. 3. $g/2\pi \gg \kappa/2\pi$: deep strong coupling: vacuum Rabi doublet split by $2g$.

Further exercises (no solution given)

(i) Einstein A/B directly from the quantised-field golden rule; (ii) numerical Bloch integration: π -pulse infidelity vs ΩT_2 ; (iii) n -photon Rabi frequency $2g\sqrt{n+1}$; collapse and revival; (iv) Kramers–Kronig reconstruction of ϵ_1 from a measured ϵ_2 .

Appendix A.1: interaction picture and the rotating frame

$$U = \exp(i\omega t |e\rangle\langle e|); \quad \tilde{H} = UH U^\dagger - \hbar\omega |e\rangle\langle e|.$$

Why the RWA removes the time dependence

Drive $-dE_0 \cos \omega t = -\frac{dE_0}{2}(e^{i\omega t} + e^{-i\omega t})$; under U , $|e\rangle\langle g|$ picks up $e^{+i\omega t}$: paired with the drive's $e^{-i\omega t}$ it becomes *static*; the other pairing oscillates at 2ω and averages away. Valid for $\Omega, \delta \ll \omega_0$: essentially always in the optical domain. Result:

$$\tilde{H} = -\hbar\delta |e\rangle\langle e| + \frac{\hbar\Omega}{2}(|e\rangle\langle g| + \text{h.c.})$$

Appendix A.2: density matrix and the master equation

General mixture: $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, $\text{Tr}\rho = 1$, $\langle O \rangle = \text{Tr}(\rho O)$.

Lindblad master equation

$$\dot{\rho} = -\frac{i}{\hbar}[\tilde{H}, \rho] + \Gamma\left(\sigma\rho\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho\}\right), \quad \sigma = |g\rangle\langle e|$$

Commutator: coherent evolution; dissipator: spontaneous emission. Pure dephasing ($\propto \sigma_z$) shortens T_2 without touching T_1 .

Optical Bloch equations from the master equation

Matrix elements:

$$\dot{\rho}_{ee} = -\Gamma\rho_{ee} - \frac{i\Omega}{2}(\rho_{ge} - \rho_{eg}), \quad \dot{\rho}_{eg} = -\left(\frac{\Gamma}{2} - i\delta\right)\rho_{eg} - \frac{i\Omega}{2}(\rho_{ee} - \rho_{gg})$$

Substituting (u, v, w) reproduces the Bloch equations, with $T_1 = 1/\Gamma$, $T_2 = 2/\Gamma$ for purely radiative relaxation.

Appendix A.3: counting field modes

The counting used throughout

Box of side L , periodic boundary conditions: allowed $\mathbf{k}$ on a grid of spacing $2\pi/L$; modes with $|\mathbf{k}| < k$ (two polarizations): $\mathcal{N} = 2 \cdot \frac{4}{3}\pi k^3 / (2\pi/L)^3$. Differentiate, use $\omega = ck$:

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

per unit volume and angular frequency: the input to spontaneous emission (L5) and Planck (L4). The identical counting with $E(\mathbf{k})$ gives the joint density of states of L8: one technique, two chapters.

Appendix B: constants and useful relations

Quantity	Symbol	Value
Reduced Planck	$\hbar$	$1.055e-34 \text{ J s}$
Speed of light	c	$2.998e8 \text{ m/s}$
Vacuum permittivity	ϵ_0	$8.854e-12 \text{ F/m}$
Electron charge	e	$1.602e-19 \text{ C}$
Electron mass	m_e	$9.109e-31 \text{ kg}$
Bohr radius	a_0	$5.29e-11 \text{ m}$
Atomic dipole unit	ea_0	$8.48e-30 \text{ C m (2.54 D)}$
Fine structure	α	$1/137.04$
Boltzmann	k_B	$1.381e-23 \text{ J/K}$

Handy conversions

$$1 \text{ eV} \leftrightarrow 1240 \text{ nm} \leftrightarrow 2.42e14 \text{ Hz} \leftrightarrow 8066 \text{ cm}^{-1};$$

$$E_0 = \sqrt{2I/\epsilon_0 c};$$

$$1 \text{ mW in } 1 \mu\text{m waist} \approx 6e8 \text{ W/m}^2;$$

$$k_B T \text{ at } 300 \text{ K} \approx 25 \text{ meV}.$$

Appendix C: Schrödinger equation and stationary states

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle; \quad H |n\rangle = E_n |n\rangle \Rightarrow |\psi(t)\rangle = |n\rangle e^{-iE_n t/\hbar}$$

Stationary states carry only a rotating phase; the E_n are the discrete levels whose differences give spectral lines. Superposition: $|\psi\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle$: interference between these phases is the origin of every dynamical effect here.

The two-level atom in this language

$|\psi\rangle = c_g |g\rangle + c_e |e\rangle$, $|c_g|^2 + |c_e|^2 = 1$: the pair (c_g, c_e) is what the Bloch vector represents; the field makes them exchange amplitude: absorption and emission at the level of amplitudes.

Appendix D: perturbation theory

Stationary:

$$E_n = E_n^{(0)} + \langle n | V | n \rangle + \sum_{m \neq n} \frac{|\langle m | V | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} + \dots$$

states “repel” through their coupling: the light shift (L2) and the Lamb shift (L0/L5) are instances.

Time-dependent: first order for a system starting in $|i\rangle$:

$$c_f^{(1)}(t) = \frac{1}{i\hbar} \int_0^t \langle f | V(t') | i \rangle e^{i\omega_{fi}t'} dt'$$

$|c_f^{(1)}|^2$ summed over a continuum $\rightarrow$ Fermi's golden rule (L4).

When it fails

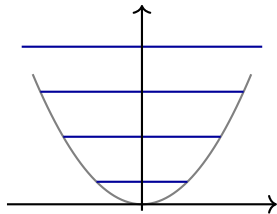
Strong resonant drive: probability sloshes fully: not a failure of physics but a different question: the non-perturbative Rabi problem, solved exactly in L2.

Appendix E: harmonic oscillator and second quantisation

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p \right), \quad [a, a^\dagger] = 1, \quad H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$, $a |n\rangle = \sqrt{n} |n-1\rangle$; zero-point energy $\frac{1}{2}\hbar\omega \rightarrow$ vacuum fluctuations.

Second quantisation of the field: one oscillator per mode $(\mathbf{k}, \lambda)$; quanta = photons; $-\mathbf{d} \cdot \mathbf{E}$ connects $|e, n\rangle \rightarrow |g, n+1\rangle$ (emission) and $|g, n-1\rangle$ (absorption): the machinery of L5 and L7.



Equally spaced levels ($\hbar\omega$); the lowest sits $\frac{1}{2}\hbar\omega$ above the well bottom.

Appendix F: the course on one page

Dipole interaction

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}$$

Lorentz susceptibility

$$\chi = \frac{Ne^2/m_e \epsilon_0}{\omega_0^2 - \omega^2 - i\gamma_c \omega}$$

Fermi golden rule

$$W = \frac{2\pi}{\hbar} |V_{fi}|^2 \rho(E_f)$$

Einstein relation

$$A/B = \hbar \omega_0^3 / \pi^2 c^3$$

Vacuum field per mode

$$\mathcal{E}_k = \sqrt{\hbar \omega_k / 2 \epsilon_0 V}$$

Spontaneous emission

$$\Gamma = \omega_0^3 d^2 / 3 \pi \epsilon_0 \hbar c^3$$

Resonant cross section

$$\sigma_0 = 3 \lambda_0^2 / 2 \pi$$

Rabi oscillation

$$P_e = (\Omega^2 / \Omega_R^2) \sin^2(\Omega_R t / 2)$$

Saturation

$$\rho_{ee} = \frac{1}{2} s / (1 + s), \quad s = I / I_{\text{sat}}$$

Light shift

$$U_{\text{dip}} = \hbar \Omega^2 / 4 \delta$$

Susceptibility (emitters)

$$\chi'' \propto \frac{1/T_2}{\delta^2 + 1/T_2^2}$$

Jaynes–Cummings

$$g = d \sqrt{\omega_c / 2 \hbar \epsilon_0 V}; \text{ splitting } 2 \hbar g \sqrt{n+1}$$

Purcell factor

$$F_P = \frac{3}{4\pi^2} (\lambda/n)^3 Q/V$$

Cooperativity

$$C = g^2 / \kappa \gamma$$

Kramers–Kronig

$$\epsilon_1 - 1 = \frac{2}{\pi} \mathcal{P} \int \frac{\omega' \epsilon_2}{\omega'^2 - \omega^2} d\omega'$$

Direct-gap edge

$$\alpha \propto (\hbar \omega - E_g)^{1/2}$$

Plasma frequency

$$\omega_p = \sqrt{Ne^2 / \epsilon_0 m^*}$$

References, and how to use these notes

- ▶ Cohen-Tannoudji, Dupont-Roc, Grynberg, *Atom-Photon Interactions* (L1–L3, appendix)
- ▶ Loudon, *The Quantum Theory of Light* (L3–L5)
- ▶ Fox, *Quantum Optics* and *Optical Properties of Solids* (M1 level; L8)
- ▶ Haroche & Raimond, *Exploring the Quantum* (L7)
- ▶ Grynberg, Aspect, Fabre, *Introduction to Quantum Optics*
- ▶ Kittel; Yu & Cardona (solids, excitons, polaritons)

How to use the notes

Read each lecture once for the narrative; then reproduce every orange box with pen and paper and redo every green box with the constants of Appendix B. The blue boxes and the Appendix-F sheet are the minimum to remember.